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Research article

Several integral inequalities for (α, s, m) -convex functions

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Abstract: In this paper, we establish several new integral inequalities for (α, s, m)-convex functions. We recapture the Hermite-Hadamard inequality as a particular case. In order to obtain our results, we use classical inequalities such as Hölder inequality, Hölder-Işcan inequality and Power mean inequality. We formulate several bounds involving special functions like classical Euler-Gamma, Beta and Psi-Gamma functions. We also give some applications.

Keywords: convex function; (α, s, m)-convex function; Hermite-Hadamard inequality; Riemann-Liouville fractional integrals; Hölder's inequality; power mean inequality; Psi-Gamma functions
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1. Introduction

The general theory of convex functions is the origin of powerful tools for the study of problems in analysis. Convex functions are thoroughly associated to the theory of inequalities. Inequalities involving convex functions are the most efficient tools in the development of several branches of mathematics and has been given considerable attention in the literature (see [1, 2]) and references therein). We start by giving some renowned definitions:

Definition 1.1. *A function $f : I \subseteq R \to R$ is said convex if*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$.

In [3] Toader defines the $m-$convexity:

Definition 1.2. *A function* $f:[0,d]\to R_0=[0,\infty)$ *is said* $m-$*convex on* $[0,d]$ *for some* $m\in(0,1]$, *if*

$$f(ta+m(1-t)b)\le tf(a)+m(1-t)f(b),\quad \forall a,b\in[0,d] \text{ and } t\in[0,1].$$

In [4] Miheşan introduced the following class of functions:

Definition 1.3. *A function* $f:[0,d]\to R_0=[0,\infty)$ *is said* $(\alpha,m)-$ *convex on* $[0,d]$, $(\alpha,m)\in(0,1]^2$, *if*

$$f(ta+m(1-t)b)\le t^{\alpha}f(a)+m(1-t^{\alpha})f(b),\quad \forall a,b\in[0,d] \text{ and } t\in[0,1].$$

Let K_{α}^{m} *be the set of* $(\alpha,m)-$ *convex functions on* $[0,d]$.

In [5] Park J. introduced the following class of $(s,m)-$convex functions in the second sense:

Definition 1.4. *For some fixed* $s\in(0,1]$ *and* $m\in[0,1]$ *a mapping* $f:[0,d]\to\mathbb{R}$ *is said* $(s,m)-$ *convex in the second sense on* $[0,d]$, *if*

$$f(ta+m(1-t)b)\le t^{s}f(a)+m(1-t)^{s}f(b)$$

holds for all $a,b\in[0,d]$ *and* $t\in[0,1]$.

In [6], Özdemir et al. gave the following lemmas for twice differentiable functions.

Lemma 1.1. *Let* $f:I\subseteq R\to R$ *be a twice differentiable mapping on* I°*(interior of* I*),* $a\neq b\in I$ *and* $f''\in L[a,b]$, *then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)\,dt.$$

Lemma 1.2. *Let* $f:I\subseteq R\to R$ *be a twice differentiable mapping on* I°, *where* $a,b\in I$ *with* $a<mb$ *and* $m\in(0,1]$. *If* $f''\in L[a,b]$, *then the following equality holds:*

$$\begin{aligned}&\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx \qquad (1.1)\\ &=\frac{(mb-a)^2}{2}\int_0^1 t(1-t)f''(ta+m(1-t)b)\,dt.\end{aligned}$$

In [7], M. Z. Sarıkaya and N. Aktan obtained the following Trapezoid inequality for convex functions

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{(b-a)^2}{24}\Big[|f''(a)|+|f''(b)|\Big]. \qquad (1.2)$$

In [8], I. Işcan gave a refinement of the Hölder's integral inequality as follows:

Theorem 1.1. *(Hölder- İşcan integral inequality) Let* $p>1$ *and* $q^{-1}+p^{-1}=1$. *If* f *and* g *are real functions defined on interval* $[a,b]$ *and if* $|f|^p$ *and* $|g|^q$ *are integrable functions on* $[a,b]$, *then*

$$\begin{aligned}\int_a^b|f(t)g(t)|\,dt\le&\ \frac{1}{b-a}\left(\int_a^b(b-t)|f(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b(b-t)|g(t)|^q\,dt\right)^{\frac{1}{q}}\\ &+\frac{1}{b-a}\left(\int_a^b(t-a)|f(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b(t-a)|g(t)|^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

An refinement of power-mean integral inequality as a different version of the Hölder- İşcan integral inequality can be given as follows:

Theorem 1.2. *(Improved power-mean integral inequality [9]) Let* $q \geq 1$*. If* f *and* g *are real functions defined on* $[a,b]$ *and if* $|f|$, $|f|\,|g|^q$ *are integrable functions on* $[a,b]$*, then*

$$\begin{aligned}\int_a^b |f(t)\,g(t)|\,dt \quad &\leq \quad \frac{1}{b-a}\left(\int_a^b (b-t)\,|f(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b (b-t)\,|f(t)|\,|g(t)|^q\,dt\right)^{\frac{1}{q}}\\ &\quad +\frac{1}{b-a}\left(\int_a^b (t-a)\,|f(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b (t-a)\,|f(t)|\,|g(t)|^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

There is a massive literature on Hermite – Hadamard type inequalities via different kind of convexities. The most notable are m–convex, (α,m) –convex, s –convex and extended s– convex (see [10–19]). Besides this, we mention recent results related to the Hermite–Hadamard type inequalities, for example, see [20–22] and the references therein. In the next section we consider the most recent generalized convex functions and its properties which covers all the above as particular cases.

2. Some properties of (α, s, m)–convex functions

Definition 2.1. *[23] The function* $f : [0,d] \to R$ *is said to be* (α,s,m) *–convex, if we have*

$$f(ta+m(1-t)b) \leq t^{\alpha s} f(a) + m(1-t^{\alpha})^s f(b),$$

where $a,b \in [0,d]$ *,* $t \in (0,1)$ *and for some* $s \in [-1,1]$ *,* $(\alpha,m) \in (0,1]^2$ *.*

Here $K_{\alpha}^{s,m}$ *is the set of* (α,s,m) *convex functions on* $[0,d]$ *.*

Remark 2.1. *(i) If* $s = 1$*, then* f *is an* (α,m) *- convex function on* $(0,d]$*.*

(ii) If $\alpha = 1$*, then* f *is an extended* (s,m) *- convex function on* $(0,d]$*.*

(iii) If $\alpha = m = 1$*, then* f *is an extended* s *-convex function on* $(0,d]$*.*

(iv) If $\alpha = s = m = 1$*, then f is a convex function on* $(0,d]$*.*

Proposition 2.1. *If a function* f *is* (α,m) *– convex for all* $s \in [-1,1]$*, then it is also* (α,s,m) *– convex.*

Proof. Since $f \in K_{\alpha}^{m}$, we have

$$f(ta+m(1-t)b) \leq t^{\alpha} f(a) + m(1-t^{\alpha}) f(b)$$

$\forall\ a,b \in [0,d]$, $t \in [0,1]$ and $(\alpha,m) \in (0,1]^2$. On the other hand, we have $t^{\alpha} \leq t^{\alpha s}$ and $1-t^{\alpha} \leq (1-t^{\alpha})^s$, for $s \in [-1,1]$, thus

$$f(ta+m(1-t)b) \leq t^{\alpha} f(a) + m(1-t^{\alpha}) f(b) \leq t^{\alpha s} f(a) + m(1-t^{\alpha})^s f(b),$$

i.e $f \in K_{\alpha}^{s,m}$. □

Proposition 2.2. *Let* $\alpha \in (0,1]$ *and* $s \in [-1,1]$, *then*

(i) *when* $s \in (-1,1]$, *we have*

$$M_\beta(\alpha,s) \approx \int_0^1 t(1-t)(1-t^\alpha)^s\,dt = \frac{1}{\alpha}\left\{\beta\left(\frac{2}{\alpha}, s+1\right) - \beta\left(\frac{3}{\alpha}, s+1\right)\right\},$$

(ii) *when* $s=-1$, *we have* $\int_0^1 \frac{t(1-t)}{(1-t^\alpha)}dt = \frac{1}{\alpha}\left\{\Psi\left(\frac{3}{\alpha}\right) - \Psi\left(\frac{2}{\alpha}\right)\right\}$,

where $\Gamma(.)$, $\beta(.,.)$, $\Psi(.)$ *are the classical Euler-Gamma, Beta and Psi-Gamma functions, respectively.*

$$\Gamma(x) = \int_0^\infty z^{x-1}e^z dz, \quad \beta(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}dz \;(x>0,\; y>0),$$
$$\Psi(x) = \frac{d\ln\Gamma(x)}{dx} = -\int_0^\infty \frac{1}{1-e^{-z}}e^{-zx}dz.$$

Proof. (i)

$$M_\beta(\alpha,s) \approx \int_0^1 t(1-t)(1-t^\alpha)^s\,dt.$$

If we change the variable $t^\alpha = z \Rightarrow t = z^{\frac{1}{\alpha}}$, we get $dt = \frac{1}{\alpha}z^{\frac{1}{\alpha}-1}dz$. Hence

$$\begin{aligned}\int_0^1 z^{\frac{1}{\alpha}}(1-z)^s\frac{1}{\alpha}z^{\frac{1}{\alpha}-1}dz \;&-\; \int_0^1 z^{\frac{2}{\alpha}}(1-z)^s\frac{1}{\alpha}z^{\frac{1}{\alpha}-1}dz \\ &= \frac{1}{\alpha}\int_0^1 z^{\frac{2}{\alpha}-1}(1-z)^s dz - \frac{1}{\alpha}\int_0^1 z^{\frac{3}{\alpha}-1}(1-z)^s dz \\ &= \frac{1}{\alpha}\left\{\beta\left(\frac{2}{\alpha}, s+1\right) - \beta\left(\frac{3}{\alpha}, s+1\right)\right\},\end{aligned}$$

which completes the proof of proposition (i).

(ii) When $s=-1$

$$\int_0^1 t(1-t)(1-t^\alpha)^{-1}dt = \int_0^1 \frac{t(1-t)}{1-t^\alpha}dt.$$

Again using the change of variable $t^\alpha = z \Rightarrow t = z^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}\int_0^1 \frac{t(1-t)}{1-t^\alpha}dt &= \int_0^1 \frac{z^{\frac{1}{\alpha}} - z^{\frac{2}{\alpha}}}{1-z}\frac{1}{\alpha}z^{\frac{1}{\alpha}-1}dz = \frac{1}{\alpha}\int_0^1 \frac{\left(1-z^{\frac{3}{\alpha}-1}\right) - \left(1-z^{\frac{2}{\alpha}-1}\right)}{1-z}dz \\ &= \frac{1}{\alpha}\int_0^1 \frac{\left(1-z^{\frac{3}{\alpha}-1}\right)}{1-z}dz - \frac{1}{\alpha}\int_0^1 \frac{\left(1-z^{\frac{2}{\alpha}-1}\right)}{1-z}dz = \frac{1}{\alpha}\left\{\left[\Psi\left(\frac{3}{\alpha}\right)+\gamma\right] - \left[\Psi\left(\frac{2}{\alpha}\right)+\gamma\right]\right\} \\ &= \frac{1}{\alpha}\left\{\left[\Psi\left(\frac{3}{\alpha}\right)\right] - \left[\Psi\left(\frac{2}{\alpha}\right)\right]\right\},\end{aligned}$$

where $\Psi(x) + \gamma = \int_0^1 \frac{1-t^{x-1}}{1-t}dt$ and $\gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t}\right)\frac{1}{t}\,dt$ is Euler–Mascheroini constant (see p.258 [24]). Which completes the proof of proposition (ii). □

3. Hermite-Hadamard type integral inequalities for (α, s, m)-convex functions

Theorem 3.1. *If* $f : [0, d] \to R$ *is* $(\alpha, s, m)-$*convex, we have*

$$f\left(\frac{a+mb}{2}\right) \leq \frac{1}{mb-a}\int_a^{mb} f(t)\,dt \leq \frac{1}{\alpha s+1}\left(f(a)+mf(b)\right), \tag{3.1}$$

where $a, b \in [0, d]$, $a < mb$ *and* $for\ some\ \ s \in [0, 1]$, $(\alpha, m) \in (0, 1]^2$.

Proof. The left-hand side of (3.1) is easy to prove:

$$\begin{aligned}\frac{1}{mb-a}\int_a^{mb} f(t)\,dt &= \frac{1}{mb-a}\left(\int_a^{\frac{a+mb}{2}} f(t)\,dt + \int_{\frac{a+mb}{2}}^{mb} f(t)\,dt\right)\\ &= \frac{1}{2}\int_0^1\left[f\left(\frac{a+mb-t(mb-a)}{2}\right)+f\left(\frac{a+mb+t(mb-a)}{2}\right)\right]dt\\ &\geq f\left(\frac{a+mb}{2}\right).\end{aligned}$$

The proof of the right-hand side (3.1).
By the $(\alpha, s, m)-$convexity of f we observe that

$$f(ta+m(1-t)b) \leq t^{\alpha s}f(a)+m(1-t^{\alpha})^s f(b), \quad \forall t \in [0, 1].$$

Integrating the resulting inequality with respect to t, we get

$$\int_0^1 f(ta+m(1-t)b)dt \leq \int_0^1 \left[t^{\alpha s}f(a)+m(1-t^{\alpha})^s f(b)\right]dt$$

as

$$\int_0^1 f(ta+m(1-t)b)dt = \frac{1}{mb-a}\int_a^{mb} f(t)\,dt$$

and

$$\begin{aligned}\int_0^1 \left[t^{\alpha s}f(a)+m(1-t^{\alpha})^s f(b)\right]dt &= \int_0^1 t^{\alpha s}f(a)dt + \int_0^1 m(1-t^{\alpha})^s f(b)dt\\ &\leq \int_0^1 t^{\alpha s}f(a)dt + mf(b)\int_0^1 (1-t)^{\alpha s}dt\\ &= \frac{1}{\alpha s+1}\left(f(a)+mf(b)\right),\end{aligned}$$

which proved the theorem as we used the fact

$$\int_0^1 (1-t^{\alpha})^s dt \leq \int_0^1 (1-t)^{\alpha s}dt = \frac{1}{\alpha s+1}.$$

□

Remark 3.1. *If we take $\alpha = m = s = 1$. The function f becomes $(1,1,1)-$convex. Hence the inequality* (3.1) *reduces to the Hermite- Hadamard inequality :*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{2}(f(a)+f(b)).$$

Theorem 3.2. *Let $f : I \subset (0,d] \to R$ be a differentiable function on I°(I° is interior of I) for $a,b \in I^\circ$ such that $f'' \in L[a,b]$ with $0 < a < mb$. If $|f''|$ is an $(\alpha,s,m)-$convex function on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx\right| \le \frac{(mb-a)^2}{2}\frac{|f''(a)| + m|f''(b)|}{(\alpha s+2)(\alpha s+3)}, \tag{3.2}$$

where $(\alpha, m) \in (0,1]^2$, $s \in (0,1]$.

Proof. From Lemma 1.2 and using the $(\alpha,s,m)-$convexity of $|f''|$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx\right| \\
\le\ & \frac{(mb-a)^2}{2}\int_0^1 t(1-t)\left[t^{\alpha s}|f''(a)| + m(1-t^\alpha)^s|f''(b)|\right]dt \\
=\ & \frac{(mb-a)^2}{2}\left[|f''(a)|\int_0^1 t^{\alpha s+1}(1-t)\,dt + m|f''(b)|\int_0^1 t(1-t)(1-t^\alpha)^s\,dt\right] \\
\le\ & \frac{(mb-a)^2}{2}\left[|f''(a)|\int_0^1 t^{\alpha s+1}(1-t)\,dt + m|f''(b)|\int_0^1 t(1-t)(1-t)^{\alpha s}\,dt\right] \\
=\ & \frac{(mb-a)^2}{2}\frac{|f''(a)| + m|f''(b)|}{(\alpha s+2)(\alpha s+3)},
\end{aligned}$$

where we used the facts that, since $t^\alpha \ge t$ for all $\alpha \in (0,1]$ and $t \in [0,1]$, we have $-t^\alpha \le -t$ or $1 - t^\alpha \le 1 - t$, $1 - t^\alpha \le (1-t)^\alpha$and $(1-t^\alpha)^s \le (1-t)^{\alpha s}$, $s \in (0,1]$. If we multiply the resulting inequality with $t(1-t)$ and taking integral on $[0,1]$, we have

$$\int_0^1 t(1-t)(1-t^\alpha)^s\,dt \le \int_0^1 t(1-t)(1-t)^{\alpha s}\,dt = \frac{1}{(\alpha s+2)(\alpha s+3)}$$

and

$$\int_0^1 t^{\alpha s+1}(1-t)\,dt = \frac{1}{(\alpha s+2)(\alpha s+3)}.$$

□

Corollary 3.1. *From Remark 2.1 (iv), since f(.) is a convex function on $[a,b]$ for $m = \alpha = s = 1$, we obtain the inequality* (1.2).

Corollary 3.2. *Under the same conditions in Theorem 3.2 with $|f''(x)| \le M$ for all $x \in [a,b]$, we have*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx\right| \le M\frac{(mb-a)^2}{2} \times \frac{(m+1)}{(\alpha s+2)(\alpha s+3)}.$$

The following Theorem presents a upper bound for (α, s, m)-convex functions.

Theorem 3.3. *Let $f : I \subset (0,d] \to R$ be a differentiable function on $I^\circ (I^\circ$ is interior of I) for $a, b \in I^\circ$, $f'' \in L[a,b]$, $(\alpha, m) \in (0,1]^2$, $s \in (-1,1]$ with $0 < a < mb$. If $|f''|^q$ is an (α, s, m) −convex function on $[a,b]$ for $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.3}$$
$$\leq \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}} \times \left[\frac{|f''(a)|^q}{(\alpha s+2)(\alpha s+3)} + mM_\beta(\alpha,s)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}},$$

where $M_\beta(\alpha, s)$ is given in Proposition 2.2(i).

Proof. First, we assume that $q = 1$. From Lemma 1.1 and using the (α, s, m) −convexity with properties of modulus we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''\left(ta+m(1-t)\frac{b}{m}\right)\right|dt\\ &\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left(t^{\alpha s}|f''(a)| + m(1-t^\alpha)^s\left|f''\left(\frac{b}{m}\right)\right|\right)dt\\ &= \frac{(b-a)^2}{2}\left[\frac{|f''(a)|}{(\alpha s+2)(\alpha s+3)} + mM_\beta(\alpha,s)\left|f''\left(\frac{b}{m}\right)\right|\right],\end{aligned}$$

which completes the proof for q = 1. Here we also used the Proposition 2.2 (i).

Secondly, suppose now that $q > 1$. From Lemma 1.1 and using the Hölder's integral inequality for $q > 1$ with properties of modulus, we have

$$\int_0^1 \left(t-t^2\right)\left|f''\left(ta+m(1-t)\frac{b}{m}\right)\right|dt = \int_0^1 \left(t-t^2\right)^{1-\frac{1}{q}}\left(t-t^2\right)^{\frac{1}{q}}\left|f''\left(ta+m(1-t)\frac{b}{m}\right)\right|dt$$
$$\leq \left[\int_0^1 \left(t-t^2\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 \left(t-t^2\right)\left|f''\left(ta+m(1-t)\frac{b}{m}\right)\right|^q dt\right]^{\frac{1}{q}}, \tag{3.4}$$

where $p^{-1} + q^{-1} = 1$.

Since $|f''|^q$ is (α, s, m)−convex on $[a, b]$, we know that for every $t \in [0, 1]$.

$$\left|f''(ta+m(1-t)\frac{b}{m})\right|^q \leq t^{\alpha s}|f''(a)|^q + m(1-t^\alpha)^s\left|f''(\frac{b}{m})\right|^q. \tag{3.5}$$

From (3.4), (3.5) and Proposition 2.2 (i)

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{2}\left[\int_0^1 \left(t-t^2\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 \left(t-t^2\right)\left|f''\left(ta+m(1-t)\frac{b}{m}\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^2}{2}\left[\int_0^1\left(t-t^2\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1\left(t-t^2\right)\left(\begin{array}{c} t^{\alpha s}\left|f''(a)\right|^q \\ +m(1-t^\alpha)^s\left|f''(\frac{b}{m})\right|^q\end{array}\right)dt\right]^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\frac{\left|f''(a)\right|^q}{(\alpha s+2)(\alpha s+3)}+mM_\beta(\alpha,s)\left|f''(\frac{b}{m})\right|^q\right]^{\frac{1}{q}},$$

which completes the inequality (3.3). □

Corollary 3.3. *Under the same conditions in Theorem 3.3 with* $|f''(x)| \leq M$ *and* $m = 1$ *for all* $x \in [a,b]$*, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq M\frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\frac{1}{(\alpha s+2)(\alpha s+3)}+M_\beta(\alpha,s)\right]^{\frac{1}{q}}.$$

Theorem 3.4. *When* $s = -1$*, under the same conditions of Theorem 3.3, we obtain the following result*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}} \tag{3.6}$$
$$\times\left[\frac{\left|f''(a)\right|^q}{(2-\alpha)(3-\alpha)}+m\left|f''\left(\frac{b}{m}\right)\right|^q\frac{1}{\alpha}\left\{\Psi\left(\frac{3}{\alpha}\right)-\Psi\left(\frac{2}{\alpha}\right)\right\}\right]^{\frac{1}{q}}.$$

Proof. Using the (α,s,m) −convexity of $|f''|^q$ and Proposition 2.2(ii) with the properties of modulus, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^2}{2}\left[\int_0^1\left(t-t^2\right)\right]^{1-\frac{1}{q}} \times\left[\int_0^1\left(t-t^2\right)\left(t^{-\alpha}\left|f''(a)\right|^q+m(1-t^\alpha)^{-1}\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt\right]^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\int_0^1\left(t-t^2\right)\left(t^{-\alpha}\left|f''(a)\right|^q+m(1-t^\alpha)^{-1}\left|f''(\frac{b}{m})\right|^q\right)dt\right]^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left|f''(a)\right|^q\int_0^1\left(t^{1-\alpha}-t^{2-\alpha}\right)dt+m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1\frac{t-t^2}{1-t^\alpha}dt\right]^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\frac{\left|f''(a)\right|^q}{(2-\alpha)(3-\alpha)}+m\left|f''(\frac{b}{m})\right|^q\frac{1}{\alpha}\left\{\Psi\left(\frac{3}{\alpha}\right)-\Psi\left(\frac{2}{\alpha}\right)\right\}\right]^{\frac{1}{q}},$$

which completes the inequality (3.6). □

Corollary 3.4. *Under the same conditions in Theorem 3.4 with* $|f''(x)| \leq M$ *and* $m = 1$ *for all* $x \in [a,b]$*, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq M\frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\times\left[\frac{1}{(2-\alpha)(3-\alpha)}+\frac{1}{\alpha}\left\{\Psi\left(\frac{3}{\alpha}\right)-\Psi\left(\frac{2}{\alpha}\right)\right\}\right]^{\frac{1}{q}}.$$

Theorem 3.5. *Let $f : I \subset (0,d] \to R$ be a twice differentiable function on I° for $a,b \in I^\circ$ such that $f'' \in L[a,b]$, $(\alpha,m) \in (0,1]^2, s \in (-1,1]$ with $0 < a < mb$. If $|f''|^q$ is an $(\alpha,s,m)-$convex function on $[a,b]$ for $q \geq 1$ and $q \geq r > 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2}\left(\beta\left(\frac{2q-r-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}} \tag{3.7}$$

$$\times\left[\frac{|f''(a)|^q}{\alpha s+r+1} + \frac{m}{\alpha}\left|f''(\frac{b}{m})\right|^q\left(\beta\left(\frac{r+1}{\alpha},s+1\right)\right)\right]^{\frac{1}{q}}.$$

Proof. Using the Lemma 1.1, Hölder's inequality and $(\alpha,s,m)-$convexity of $|f''|^q$ we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{(b-a)^2}{2}\left[\int_0^1\left(t^{\frac{q-r}{q-1}}(1-t)^{\frac{q}{q-1}}\right)dt\right]^{1-\frac{1}{q}}\\ &\quad\times\left[\int_0^1 t^r\left(t^{\alpha s}|f''(a)|^q + m(1-t^\alpha)^s\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt\right]^{\frac{1}{q}}\\ &= \frac{(b-a)^2}{2}\left[\beta\left(\frac{2q-r-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\\ &\quad\times\left[|f''(a)|^q\int_0^1 t^{\alpha s+r}dt + m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1 t^r(1-t^\alpha)^s dt\right]^{\frac{1}{q}}.\end{aligned}$$

After simplifying we get inequality (3.7). □

An immediate consequences of Theorem 3.5 by considering special cases for $(r = 0)$ and $(m = 1)$ can be given as:

Corollary 3.5. *Under the assumptions of Theorem 3.5, the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2}\left(\beta\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}} \tag{3.8}$$

$$\times\left[\frac{|f''(a)|^q}{\alpha s+1} + \frac{m}{\alpha}\left|f''\left(\frac{b}{m}\right)\right|^q\left(\beta\left(\frac{1}{\alpha},s+1\right)\right)\right]^{\frac{1}{q}}.$$

Corollary 3.6. *Under the assumptions of Theorem 3.5, the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2}\left(\beta\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}} \tag{3.9}$$

$$\times\left[\frac{|f''(a)|^q}{\alpha s+1} + \frac{1}{\alpha}\;|f''(b)|^q\left(\beta\left(\frac{1}{\alpha},s+1\right)\right)\right]^{\frac{1}{q}}.$$

Corollary 3.7. *Under the same conditions in Theorem 3.5 with $|f''(x)| \leq M$ and $m = 1$, for all $x \in [a,b]$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq M\frac{(b-a)^2}{2}\left[\beta\left(\frac{2q-r-1}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}} \times\left[\frac{1}{\alpha s+r+1} + \frac{1}{a}\beta\left(\frac{r+1}{\alpha}, s+1\right)\right]^{\frac{1}{q}}.$$

Theorem 3.6. *Let $f: I \subset (0,d] \to R$ be a twice differentiable function on I° such that $f'' \in L[a,b]$ for $a,b \in I^\circ$, $(\alpha,m) \in (0,1]^2$, $s \in (-1,1]$ with $0 < a < mb$. If $|f''|^q$ is an (α,s,m)−convex function on $[a,b]$ for $q \geq 1$, $q \geq r > 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq K\left[\beta\left(\frac{2q-r-1}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}} \times\left[\frac{1}{\alpha s+r+1}|f''(a)|^q + \frac{m}{\alpha}\cdot\beta\left(\frac{r+1}{\alpha}, s+1\right)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}, \tag{3.10}$$

where $K = \frac{(b-a)^2}{2}$.

Proof. By using Lemma 1.1 and Hölder's inequality with (α,s,m)−convexity of $|f''|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & K\left[\int_0^1 t^{\frac{q-r}{q-1}}(1-t)^{\frac{q}{q-1}}\,dt\right]^{1-\frac{1}{q}}\left[\int_0^1 t^r\left(\begin{array}{c} t^{\alpha s}|f''(a)|^q \\ +m(1-t^\alpha)^s\left|f''\left(\frac{b}{m}\right)\right|^q\end{array}\right)dt\right]^{\frac{1}{q}} \\
\leq\ & K\left\{\left[\int_0^1 t^{\frac{q-r}{q-1}}(1-t)^{\frac{q}{q-1}}\,dt\right]^{1-\frac{1}{q}}\left[\begin{array}{c} |f''(a)|^q\int_0^1 t^{r+\alpha s}dt \\ +m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1 t^r(1-t^\alpha)^s\,dt\end{array}\right]^{\frac{1}{q}}\right\} \\
=\ & K\left\{\left[B\left(\frac{2q-r-1}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\left[\begin{array}{c} |f''(a)|^q\int_0^1 t^{r+\alpha s}dt \\ +m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1 t^r(1-t^\alpha)^s\,dt\end{array}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Here we used the fact that

$$\int_0^1 t^{\frac{q-r}{q-1}}(1-t)^{\frac{q}{q-1}}\,dt = B\left(\frac{2q-r-1}{q-1}, \frac{2q-1}{q-1}\right)$$

and with a formal change of variables $t^\alpha = z$

$$\int_0^1 t^r(1-t^\alpha)^s\,dt = \frac{1}{\alpha}\int_0^1 z^{\frac{r}{\alpha}}(1-z)^s\,dz = \frac{1}{\alpha}B\left(\frac{r+1}{\alpha},\ s+1\right).$$

If we write the above integrals places we obtain the inequality (3.10). □

Theorem 3.7. *Let $f : I \subset (0, d] \to R$ be a twice differentiable function on I° such that $f'' \in L[a, b]$ for $a, b \in I^\circ$, $(\alpha, m) \in (0, 1]^2$, $s \in [0, 1]$ with $0 < a < mb$. Also let $p > 1$ such that $q = \frac{p}{p-1}$, if $|f''|^q$ is an $(\alpha, s, m)-$ convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(t)\,dt\right| \tag{3.11}$$

$$\leq G\left\{\begin{array}{c}\left(\frac{3p}{2(p+1)(p+2)(2p+1)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q}{(\alpha s+1)(\alpha s+2)} + \frac{m|f''(b)|^q}{(\alpha s+2)}\right)^{\frac{1}{q}} \\ +\left(\frac{p}{2(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q}{(\alpha s+2)} + \frac{m\left|f''(b)\right|^q}{(\alpha s+1)(\alpha s+2)}\right)^{\frac{1}{q}}\end{array}\right\},$$

where $G = \frac{(mb-a)^2}{2}$.

Proof. Using Lemma 1.2 and Hölder- İşcan integral inequality along with (α, s, m) −convexity of $|f''|^q$, we have

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(t)\,dt\right| \leq G\int_0^1 \left(t-t^2\right)\left|f''(ta+m(1-t)b)\,dt\right|dt$$

$$\leq \quad G\left\{\begin{array}{c}\left(\int_0^1 (1-t)\left|t-t^2\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f''(ta+m(1-t)b)\,dt\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 t\left|t-t^2\right|^p dt\right)^{\frac{1}{p}}\left(t\left|f''(ta+m(1-t)b)\,dt\right|^q dt\right)^{\frac{1}{q}}.\end{array}\right\}$$

Using the fact that $|m-n|^c \leq m^c - n^c$ for $c > 1$ and for $m, n > 0$; $m > n$. The right-hand side of above inequality becomes

$$\leq \quad G\left\{\begin{array}{c}\left(\int_0^1 (1-t)\left(t^p - t^{2p}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\{t^{\alpha s}|f''(a)|^q + m|f''(b)|^q(1-t)^{\alpha s}\}\,dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 t\left(t^p - t^{2p}\right)dt\right)^{\frac{1}{p}}(t\{t^{\alpha s}|f''(a)|^q + m|f''(b)|^q(1-t)^{\alpha s}\}\,dt)^{\frac{1}{q}}.\end{array}\right\}$$

Simplifying above integrals, we obtain the result of the Theorem 3.7. □

Corollary 3.8. *Under the same conditions in Theorem 3.7 with $|f''(x)| \leq M$ for all $x \in [a, b]$, we have*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx\right| \leq G\cdot M\cdot\left\{\begin{array}{c}\left(\frac{3p}{2(p+1)(p+2)(2p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{(\alpha s+1)(\alpha s+2)} + \frac{m}{(\alpha s+2)}\right)^{\frac{1}{q}} \\ +\left(\frac{p}{2(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{(\alpha s+2)} + \frac{m}{(\alpha s+1)(\alpha s+2)}\right)^{\frac{1}{q}}\end{array}\right\}$$

Theorem 3.8. *Let $f : I \subset (0, d] \to R$ be a twice differentiable function on I° such that $f'' \in L[a, b]$ for $a, b \in I^\circ$, $(\alpha, m) \in (0, 1]^2$, $s \in [0, 1]$ with $0 < a < mb$. If $|f''|^q$ is an $(\alpha, s, m)-$convex on $[a, b]$ for $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(t)\,dt\right| \leq \frac{G}{12}\left(\frac{12}{(\alpha s+3)(\alpha s+4)}\right)^{\frac{1}{q}} \tag{3.12}$$

$$\times\left\{\left(\frac{2\left|f''(a)\right|^{q}}{\alpha s+2}+m\left|f''(b)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|f''(a)\right|^{q}+\frac{2m\left|f''(b)\right|^{q}}{\alpha s+2}\right)^{\frac{1}{q}}\right\},$$

where $G=\frac{(mb-a)^2}{2}$.

Proof. Using Lemma 1.2 and improved power mean integral inequality with (α, s, m)−convexity of $|f''|^q$, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(t)\,dz\right|$$

$$\leq G\cdot\left\{\begin{array}{c}\left(\int_0^1(1-t)\left|t-t^2\right|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|t-t^2\right|\,\left|f''(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\ +\left(\int_0^1 t\left|t-t^2\right|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|t-t^2\right|\,\left|f''(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right\}$$

By employing (α, s, m)−convexity and simplifying, we have

$$\leq\ G\cdot\left\{\begin{array}{c}\left(\int_0^1(1-t)\left|t-t^2\right|dt\right)^{1-\frac{1}{q}}\\ \times\left[\int_0^1(1-t)\left|t-t^2\right|\,\left(t^{\alpha s}\left|f''(a)\right|^q+m\left|f''(b)\right|^q(1-t^{\alpha})^s\,dt\right)\right]^{\frac{1}{q}}\\ +\left(\int_0^1 t\left|t-t^2\right|\,dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t\left|t-t^2\right|\,\left(t^{\alpha s}\left|f''(a)\right|^q+m\left|f''(b)\right|^q(1-t^{\alpha})^s\right)dt\right]^{\frac{1}{q}}\end{array}\right\}\qquad(3.13)$$

It can be noticed that

$$\int_0^1(1-t)\left|t-t^2\right|\,dt=\int_0^1 t\left|t-t^2\right|\,dt=\frac{1}{12}\qquad(3.14)$$

and

$$\int_0^1 t^{\alpha s}(1-t)\left|t-t^2\right|\,dt\ =\ \frac{2}{(\alpha s+2)(\alpha s+3)(\alpha s+4)}\qquad(3.15)$$

and

$$\begin{aligned}\int_0^1(1-t)\left|t-t^2\right|\,(1-t^{\alpha})^s\,dt\ &\leq\ \int_0^1\left|t-t^2\right|\,(1-t)^{\alpha s+1}\,dt\qquad(3.16)\\ &=\ \int_0^1 t(1-t)^{\alpha s+2}\,dt=\frac{1}{(\alpha s+3)(\alpha s+4)}\end{aligned}$$

$$\begin{aligned}\int_0^1 t\left|t-t^2\right|\,t^{\alpha s}\,dt\ &=\ \int_0^1(1-t)\ t^{\alpha s+2}\,dt\ =\frac{1}{(\alpha s+3)(\alpha s+4)}\qquad(3.17)\\ \int_0^1 t\left|t-t^2\right|\,(1-t^{\alpha})^s\,dt\ &\leq\ \int_0^1 t\left|t-t^2\right|\,(1-t)^{\alpha s}\,dt\\ &=\ \int_0^1 t^2\ (1-t)^{\alpha s+1}\,dt=\frac{2}{(\alpha s+2)(\alpha s+3)(\alpha s+4)}\end{aligned}$$

Replacing the values of the integrals computed in (3.14), (3.15), (3.16) and (3.17) in (3.13), we get (3.12). □

Corollary 3.9. *Under the same conditions in Theorem* 3.8 *with* $\left|f''(x)\right| \leq M$ *for all* $x \in [a,b]$, *we have*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx\right|$$
$$\leq M \cdot \frac{(mb-a)^2}{24}\left(\frac{12}{(\alpha s+3)(\alpha s+4)}\right)^{\frac{1}{q}}\left[\left(\frac{2}{\alpha s+2}+m\right)^{\frac{1}{q}} + \left(1+\frac{2m}{\alpha s+2}\right)^{\frac{1}{q}}\right].$$

4. Application to special means

Now we let us consider some special means for arbitrary positive real numbers c and d.

The arithmetic mean :
$A(c,d) = \frac{c+d}{2}$, $\quad c,d \in R$ *with* $c,d > 0$.
The geometric mean :
$G(c,d) = (c \cdot d)^{\frac{1}{2}}$, $\quad c,d \in R$ *with* $c,d > 0$.
The harmonic mean :
$H(c,d) = \frac{2\,c\,d}{c+d}$, $\quad c,d \in R \setminus \{0\}$.
The logarithmic mean :
$$L(c,d) = \begin{cases} c, & if \;\; c = d \\ \frac{d-c}{lnd-lnc}, & if \;\; c \neq d. \end{cases}$$
The Generalized logarithmric mean :
$$L_n(c,d) = \begin{cases} c, & if \;\; c = d, \\ \left(\frac{d^{n+1}-c^{n+1}}{(n+1)(d-c)}\right)^{1/n}, & if \;\; c \neq d, \;\; n \in Z\setminus\{-1,0\}; c,d > 0. \end{cases}$$

The Identric mean :
$$I(c,d) = \begin{cases} c, & if \;\; c = d \\ \frac{1}{e}\left(\frac{d^d}{c^c}\right)^{\frac{1}{d-c}}, & if \;\; c \neq d\;;\; c,d > 0. \end{cases}$$

Proposition 4.1. *Let* $a,b \in R$, $a < b$, $a,b > 0$, *then we have the following inequality holds*

$$\left|A\left(e^a,\, e^b\right) - L\left(e^a,\, e^b\right)\right|$$
$$\leq \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}} \cdot \left[\frac{|e^a|^q}{(\alpha s+2)(\alpha s+3)} + mM_\beta(\alpha,s)\left|e^{\frac{b}{m}}\right|^q\right]^{\frac{1}{q}}.$$

Proof. This assertion follows from Theorem 3.3 for $f(x) = e^x$ and $x \in R$. □

Proposition 4.2. *Let* $a,b \in R$, $a < b$, $a,b > 0$, *then we have the following inequality*

$$\left|A\left(\ln a^{-1},\;\ln b^{-1}\right) + \ln(I)\right|$$

$$\leq \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\frac{\left|\frac{1}{a^2}\right|^q}{(2-\alpha)(3-\alpha)}+\frac{m}{\alpha}\left|\frac{m}{b}\right|^{2q}\left\{\Psi\left(\frac{3}{\alpha}\right)-\Psi\left(\frac{2}{\alpha}\right)\right\}\right]^{\frac{1}{q}}.$$

Proof. This assertion follows from Theorem 3.4 for $f(x) = -lnx$ and $x > 0$. □

Proposition 4.3. *Let* $a, b \in R$, $a < b$ *and* $0 \notin [a, b]$, *then the following inequality holds*

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{\left|\frac{2}{a^3}\right|^q}{(2-\alpha)(3-\alpha)}+m\left|\frac{2m^3}{b^3}\right|^q.\frac{1}{\alpha}\left\{\Psi\left(\frac{3}{\alpha}\right)-\Psi\left(\frac{2}{\alpha}\right)\right\}\right]^{\frac{1}{q}}.$$

Proof. This assertion follows from Theorem 3.4 for $f(x) = \frac{1}{x}$; $x \in [a, b]$. □

Proposition 4.4. *Let* $a, b \in R$, $a < b$ *and* $0 \notin [a, b]$, *then the following inequality holds*

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{2}\left(B\left(\frac{2q-r-1}{q-1},\frac{2q-1}{q-1}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{\left|\frac{2}{a^3}\right|^q}{\alpha s+r+1}+\frac{m}{\alpha}\left|\frac{2m^3}{b^3}\right|^q B\left(\frac{r+1}{\alpha}, s+1\right)\right]^{\frac{1}{q}}.$$

Proof. This assertion follows from Theorem 3.5 for $f(x) = \frac{1}{x}$; $x \in [a, b]$. □

5. Conclusions

In many practical studies, it is necessary to evaluate the difference between the two quantities. From the point of view of programming an algorithm for solving optimization problems, classical inequalities with the smallest upper limit play an important role.

We have proved several new generalized integral inequalities involving (α, s, m) −convex functions for extended case of s where $s \in [-1, 1]$, we obtained Hermite–Hadamrad type inequalities. The most interesting case is extended case for $s = -1$, when we get connected to Psi-Gamma functions. We analyze new upper bounds involving special functions by employing different variants of Hölder inequality such as Hölder-Işcan inequality and Improved Power mean inequality.

Conflict of interest

The author declares no conflict of interest in this paper.

References

1. C. P. Niculescu, L. E. Persson, *Convex functions and their applications: A contemporary approach*, Second edition, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2018.
2. S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite-Hadamard type inequalities and applications*, RGMIA Monographs, Victoria University, 2000.
3. G. H. Toader, *Some generalisations of the convexity* Proc. Colloq. Approx. Optim. Cluj-Napoca (1985), 329–338.
4. V. G. şan, *A generalisation of the convexity, seminar on functional equations, approximation and convex.*, Cluj-Napoca, 1993.
5. J. Park, *Generalization of Ostrowski-type inequalities for differentiable real* (s, m)*-convex mappings*, Far East J. Math. Sci., **49** (2011), 157–171.
6. M. E. Özdemir, M. Ardıç, H. Önalan, *Hermite-Hadamard-type inequalities via (α,m)-convexity,* Comput. Math. Appl., **61** (2011), 2614–2620.
7. M. Z. Sarıkaya, N. Aktan, *On the generalizations of some integral inequalities and their applications,* Math. Comput. Modelling, **54** (2011), 2175–2182.
8. İ. Işcan, *New refinements for integral and sum forms of Hölder inequality*, J. Inequal. Appl., 2016, Article ID 304.
9. M. Kadakal, İ. Işcan, H. Kadakal, et al. *On improvements of some integral inequalities, Researchgate*, DOI: 10.13140/RG.2.2.15052.46724, Preprint, January, 2019.
10. S. Özcan, İ. Işcan, *Some new Hermite-Hadamard type inequalities for* $s-$*convex functions and their applications,* J. Inequalities Appl., **2019** (2019).
11. B. Bayraktar, M. Gürbuz, *On some integral inequalities for* $(s, m)-$*convex functions,* TWMS J. App. Eng. Math., **10** (2020), 288–295.
12. S. S. Dragomir, G. H. Toader, *Some inequalities for m-convex functions,* Studia Univ. Babes-Bolyai Math., **38** (1993), 21–28 .
13. M. K. Bakula, M. E. Ozdemir, J. Pečarić, *Hadamard type inequalities for m-convex and* (α, m)*-convex functions,* J. Ineq. Pure Appl. Math., **9** (2008), Art. 96.
14. B. Bayraktar, V. Kudaev, *Some new inequalities for* $(s, m)-$ *convex and* $(\alpha, m)-$*convex functions,* Bulletin Karganda University-Math., **94** (2019), 15–25.
15. M. K. Bakula, J. Pečarić, J. Perić, *Extensions of the Hermite-Hadamard inequality with applications,* Math. Inequal. Appl., **12** (2012), 899–921.
16. H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., **48** (1994), 100–111.
17. M. Alomari, M. Darus, U. S. Kırmacı, *Some inequalities of Hermite-Hadamard type for s-convex functions,* Acta Math. Sci., **31B** (2011), 1643–1652.
18. C. Ling, F. Qi, *Inequalities of Simpson type for functions whose third derivatives are extended s-convex functions and applications to means,* J. Comput. Anal. Appl., **19** (2015), 555–569.

19. B. Y. Xi, F. Qi, *Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means*, J. Nonlinear Convex Anal., **16** (2015), 873–890.
20. T. S. Du, J. G. Liao, Y. J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized $(s, m)-$preinvex functions*, J. Nonlinear Sci. Appl., **9** (2016), 3112–3126.
21. J. Liao, S. H. Wu, T. S. Du, *The Sugeno integral with respect to $\alpha-$preinvex functions*, Fuzzy Sets Syst., **379** (2020), 102–114.
22. T. S. Du, M. U. Awan, A. Kashuri, et al. *Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m; h)-preinvexity*, Appl. Anal., **2019** (2019), 1–21.
23. B. Y. Xi, D. D. Gao, F. Qi, *Integral inequalities of Hermite-Hadamard type for $(\alpha, s)-$convex and $(\alpha, s, m)-$convex functions*, 2018, Aviable from: https://hal.archives-ouvertes.fr/hal-01761678.(2018).
24. M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, New York, Dover Publications, USA, 1972.

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