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On addition and multiplication of points in a certain class of projective Klingenberg planes

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Abstract

Let (O, E, U, V) be the coordination quadruple of the projective Klingenberg plane (PK-plane) coordinated with dual quaternion ring $Q(\varepsilon) = Q + Q\varepsilon = \{x + y\varepsilon \mid x, y \in Q\}$, where Q is any quaternion ring over a field. In this paper, we define addition and multiplication of points on the line $OU = [0, 1, 0]$ geometrically, also we give the algebraic correspondences of them. Finally, we carry over some well-known properties of ordinary addition and multiplication to our definition.

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1 Introduction

Algebra and geometry are two essential subjects of mathematics and therefore relations between these subjects have been investigated since Euclides of Alexandra B.C. 325. In this paper, our aim is to construct a relation between algebraic and geometric definitions of addition and multiplication of points in projective Klingenberg planes which seem to be a generalization of ordinary projective planes. In this section, we give some required concepts from the literature.

A ring $\mathbf{R}$ with an identity element is called *local* if the set $\mathbf{I}$ of its non-units is an ideal.

A *projective plane* (see [1]) $(\mathcal{P}, \mathcal{L}, \in)$ is a system in which the elements of $\mathcal{P}$ are called points and the elements of $\mathcal{L}$ are called lines together with an incidence relation $\in$ between the points and lines such that

P1: If $P \neq Q$ and $P, Q \in \mathcal{P}$, then there is a unique line passing through P and Q (denoted by $P \vee Q$ or PQ).

P2: If $l, m \in \mathcal{L}$, then there exists at least one point on both l and m.

P3: There exist four points such that no three of them are collinear.

It is proven that there exists a unique intersection point of different lines.

A *projective Klingenberg plane* (PK-plane) (see [2, 3]) is a system $(\mathcal{P}, \mathcal{L}, \in, \sim)$ where $(\mathcal{P}, \mathcal{L}, \in)$ is an incidence structure and $\sim$ is an equivalence relation on $\mathcal{P} \cup \mathcal{L}$ (called neighboring) such that no point is neighbor to any line and the following axioms are satisfied:

PK1: If $P \nsim Q$, $P, Q \in \mathcal{P}$, then there is a unique line passing through P and Q (denoted by $P \vee Q$ or PQ).

PK2: If $l \nsim m$, $l, m \in \mathcal{L}$, then there is a unique point on both l and m (denoted by $l \wedge m$ or lm).

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PK3: There is a projective plane Π^* and an incidence structure epimorphism $\chi : \Pi \to \Pi^*$ such that $P \sim Q \Leftrightarrow \chi(P) = \chi(Q)$ and $l \sim m \Leftrightarrow \chi(l) = \chi(m)$.

A point $P \in \mathcal{P}$ is called *near* a line $g \in \mathcal{L}$ (which is denoted by $P \sim g$) iff there exists a line $h \sim g$ such that $P \in h$.

Now we give two theorems and a corollary from [2].

Theorem 1 *Let Π be a PK-plane with a canonical image Π^*. Choose a basis (O, U, V, E) consisting of four points whose image $(\chi(O), \chi(U), \chi(V), \chi(E))$ in Π^* forms a quadrangle. Let $g_\infty := UV$, $l := OE$, $W := l \wedge UV$, $\eta := \{P \in l \mid P \sim O\}$ and $R := \{P \in l \mid P \nsim W\}$. Let $0 := O$, $1 := E$. Then the points $P \in \mathcal{P}$ and the lines $g \in \mathcal{L}$ of Π get their coordinates as follows*:

If $P \nsim g_\infty$, let $P = (x, y, 1)$ where $(x, x, 1) = PV \wedge l$, $(y, y, 1) = PU \wedge l$.

If $P \sim g_\infty$, $P \nsim V$, let $P = (1, y, z)$ where $(1, z, 1) = ((PV \wedge UE) \vee O) \wedge EV$ and $(1, y, 1) = OP \wedge EV$.

If $P \sim V$, let $P = (w, 1, z)$ where $(1, 1, z) = PU \wedge l$, and $(w, 1, 1) = OP \wedge EU$ (clearly, $w, z \in \eta$).

If $g \nsim V$, then $g = [m, 1, p]$ where $(1, m, 1) = ((g \wedge g_\infty) \vee O) \wedge EV$, $(0, p, 1) = g \wedge OV$.

If $g \sim V$, $g \nsim g_\infty$, then $g = [1, u, v]$ where $(u, 1, 1) = ((g \wedge g_\infty) \vee O) \wedge EU$, $(v, 0, 1) = g \wedge OU$.

If $g \sim g_\infty$, then $g = [q, r, 1]$ where $(1, 0, q) = g \wedge OU$, $(0, 1, r) = g \wedge OV$. (Then $q, r \in \eta$.)

Then $O = (0, 0, 1)$, $U = (1, 0, 0)$, $V = (0, 1, 0)$, $E = (1, 1, 1)$, $OU = [0, 1, 0]$, $OV = [1, 0, 0]$, $UV = [0, 0, 1]$, $l = OE = [1, 1, 0]$ and a point $a \in R$ has coordinates $(a, a, 1)$. We note that $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$ if and only if $a_i - b_i \in \mathbf{I}$, for $i = 1, 2, 3$, dually for lines.

Theorem 2 *Let $\mathbf{R}$ be a local ring, and the set of the non-units is denoted by $\mathbf{I}$. The system $(\mathcal{P}, \mathcal{L}, \in, \sim)$ is a PK-plane where*

$$\mathcal{P} = \{(x, y, 1) \mid x, y \in \mathbf{R}\} \cup \{(1, y, z) \mid y \in \mathbf{R}, z \in \mathbf{I}\} \cup \{(w, 1, z) \mid w, z \in \mathbf{I}\},$$

$$\mathcal{L} = \{[m, 1, k] \mid m, k \in \mathbf{R}\} \cup \{[1, n, p] \mid n \in \mathbf{I}, p \in \mathbf{R}\} \cup \{[q, n, 1] \mid q, n \in \mathbf{I}\},$$

$$(x, y, 1) \in [m, 1, k] \quad \Leftrightarrow \quad y = xm + k,$$

$$(x, y, 1) \in [1, n, p] \quad \Leftrightarrow \quad x = yn + p,$$

$$(x, y, 1) \notin [q, n, 1],$$

$$(1, y, z) \in [m, 1, k] \quad \Leftrightarrow \quad y = m + zk,$$

$$(1, y, z) \in [q, n, 1] \quad \Leftrightarrow \quad z = q + yn,$$

$$(1, y, z) \notin [1, n, p],$$

$$(w, 1, z) \in [1, n, p] \quad \Leftrightarrow \quad w = n + zp,$$

$$(w, 1, z) \in [q, n, 1] \quad \Leftrightarrow \quad z = wq + n,$$

$$(w, 1, z) \notin [m, 1, k],$$

$$(x_1, x_2, x_3) \sim (y_1, y_2, y_3) \quad \Leftrightarrow \quad x_i - y_i \in \mathbf{I},$$

$$[a_1, a_2, a_3] \sim [b_1, b_2, b_3] \quad \Leftrightarrow \quad a_i - b_i \in \mathbf{I}.$$

Corollary 3 *If $t \in \mathbf{I}$, then $1 - t$ is a unit. Therefore*

$$
\begin{aligned}
&(x,y,1) \nsim (1,y,z), \qquad (x,y,1) \nsim (w,1,z), \qquad (1,y,z) \nsim (w,1,z),\\
&(w,1,z) \sim (u,1,t), \qquad (x,y,1) \sim (u,v,1) \quad \Leftrightarrow \quad x - u \in \mathbf{I}, \qquad y - v \in \mathbf{I},\\
&(1,y,z) \sim (1,u,t) \quad \Leftrightarrow \quad y - u \in \mathbf{I}.
\end{aligned}
$$

The PK-plane which is given in Theorem 1 is denoted with $PK_2(\mathbf{R})$ and it is called the PK-plane *coordinated with (the local ring)* $\mathbf{R}$.

Finally, we recall some definitions and theorems from [4]. Let $\mathbf{Q} = \{x_0 + x_1 i + x_2 j + x_3 k \mid x_0, x_1, x_2, x_3 \in \mathbf{F}\}$ be an arbitrary division ring over a field $\mathbf{F}$ (which is called the ring of quaternions over $\mathbf{F}$) with the operators addition and multiplication defined by

$$
\begin{aligned}
&(x_0 + x_1 i + x_2 j + x_3 k) + (y_0 + y_1 i + y_2 j + y_3 k)\\
&\quad = (x_0 + y_0) + (x_1 + y_1)i + (x_2 + y_2)j + (x_3 + y_3)k,\\
&(x_0 + x_1 i + x_2 j + x_3 k) \cdot (y_0 + y_1 i + y_2 j + y_3 k)\\
&\quad = (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3) + (x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2)i\\
&\qquad + (x_0 y_2 + x_2 y_0 + x_3 y_1 - x_1 y_3)j + (x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1)k.
\end{aligned}
$$

For detailed information about quaternions, see [5].

We consider the set $\mathbf{Q}(\varepsilon) = \mathbf{Q} + \mathbf{Q}\varepsilon = \{a + b\varepsilon \mid a, b \in \mathbf{Q}\}$ together with the operations

$$
\begin{aligned}
&(a + b\varepsilon) + (c + d\varepsilon) = (a + c) + (b + d)\varepsilon,\\
&(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon.
\end{aligned}
$$

Then the elements of $\mathbf{Q}(\varepsilon)$ are called *dual quaternions*.

Theorem 4 *The non-unit elements of $\mathbf{Q}(\varepsilon)$ are in the form $b\varepsilon$, for $b \in \mathbf{Q}$ and if $a \neq 0$, $a, b \in \mathbf{Q}$, $a + b\varepsilon$ is a unit and $(a + b\varepsilon)^{-1} = a^{-1} - a^{-1}ba^{-1}\varepsilon$.*

Theorem 5 *The set of non-units $\mathbf{I} = \mathbf{Q}\varepsilon = \{b\varepsilon \mid b \in \mathbf{Q}\}$ is an ideal of $\mathbf{Q}(\varepsilon)$.*

Corollary 6 *The following properties are valid*:

(i) *$\mathbf{Q}(\varepsilon)$ is a local ring* (*and it is called the dual local ring on* $\mathbf{Q}$).

(ii) *From Theorem 2, $PK_2(\mathbf{Q}(\varepsilon)) = (\mathcal{P}, \mathcal{L}, \in, \sim)$ is a PK-plane, where*

$$
\begin{aligned}
\mathcal{P} = &\left\{(x_1 + x_2\varepsilon, y_1 + y_2\varepsilon, 1) \mid x_1, x_2, y_1, y_2 \in \mathbf{Q}\right\}\\
&\cup \left\{(1, y_1 + y_2\varepsilon, z_2\varepsilon) \mid y_1, y_2, z_2 \in \mathbf{Q}\right\} \cup \left\{(w_2\varepsilon, 1, z_2\varepsilon) \mid w_2, z_2 \in \mathbf{Q}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L} = &\left\{[m_1 + m_2\varepsilon, 1, k_1 + k_2\varepsilon] \mid m_1, m_2, k_1, k_2 \in \mathbf{Q}\right\}\\
&\cup \left\{[1, n_2\varepsilon, p_1 + p_2\varepsilon] \mid n_2, p_1, p_2 \in \mathbf{Q}\right\} \cup \left\{[q_2\varepsilon, n_2\varepsilon, 1] \mid q_2, n_2 \in \mathbf{Q}\right\}.
\end{aligned}
$$

Theorem 7 *Neighbor relation $\sim$ is an equivalence relation over $\mathcal{P}$ and $\mathcal{L}$ in $PK_2(\mathbf{Q}(\varepsilon))$.*

Theorem 8 *In $PK_2(\mathbf{Q}(\varepsilon))$, the following properties are satisfied*:

(i) $(x_1 + x_2\varepsilon, y_1 + y_2\varepsilon, 1) \in [m_1 + m_2\varepsilon, 1, k_1 + k_2\varepsilon] \Leftrightarrow y_1 = x_1m_1 + k_1$, $y_2 = x_2m_1 + x_1m_2 + k_2$.

(ii) $(x_1 + x_2\varepsilon, y_1 + y_2\varepsilon, 1) \in [1, n_2\varepsilon, p_1 + p_2\varepsilon] \Leftrightarrow x_1 = p_1$, $x_2 = y_1n_2 + p_2$.

(iii) $(1, y_1 + y_2\varepsilon, z_2\varepsilon) \in [m_1 + m_2\varepsilon, 1, k_1 + k_2\varepsilon] \Leftrightarrow y_1 = m_1$, $y_2 = m_2 + z_2k_1$.

(iv) $(1, y_1 + y_2\varepsilon, z_2\varepsilon) \in [q_2\varepsilon, n_2\varepsilon, 1] \Leftrightarrow z_2 = q_2 + y_1n_2$.

(v) $(w_2\varepsilon, 1, z_2\varepsilon) \in [1, n_2\varepsilon, p_1 + p_2\varepsilon] \Leftrightarrow w_2 = n_2 + z_2p_1$.

(vi) $(w_2\varepsilon, 1, z_2\varepsilon) \in [q_2\varepsilon, n_2\varepsilon, 1] \Leftrightarrow z_2 = n_2$.

(vii) $(a_1 + a_2\varepsilon, b_1 + b_2\varepsilon, 1) \sim (c_1 + c_2\varepsilon, d_1 + d_2\varepsilon, 1) \Leftrightarrow c_1 = a_1 \wedge d_1 = b_1$.

(viii) $(1, a_1 + a_2\varepsilon, b_2\varepsilon) \sim (1, c_1 + c_2\varepsilon, d_2\varepsilon) \Leftrightarrow c_1 = a_1$.

(ix) *For every* $a_2, b_2, c_2, d_2 \in \mathbf{Q}$ $(a_2\varepsilon, 1, b_2\varepsilon) \sim (c_2\varepsilon, 1, d_2\varepsilon)$.

2 Addition and multiplication of points on the line OU in $PK_2(\mathbf{Q}(\boldsymbol{\varepsilon}))$

In this section we give the definition of addition and multiplication of points on the line OU, and also we give some useful results for calculating addition and multiplication of points where (O, U, V, E) is a base of $PK_2(\mathbf{Q}(\varepsilon))$.

Definition 9 Let A and B be non-neighbor points of $PK_2(\mathbf{Q}(\varepsilon))$ on the line OU. Then

(i) $A + B$ is defined as the intersection point of the lines LV and OU where $L = KU \wedge BS$, $K = AV \wedge OS$, $S = (1, 1, 0)$.

(ii) $A \cdot B$ is defined as the intersection point of the lines VN and OU where $N = AS \wedge OM$, $M = BV \wedge 1S$, $S = (1, 1, 0)$, $1 = (1, 0, 1)$.

Now we state a theorem which interprets Definition 9 algebraically.

Theorem 10 *Let $A = (a_1 + a_2\varepsilon, 0, 1)$ and $B = (b_1 + b_2\varepsilon, 0, 1)$ be two non-neighbor points on the line OU and let $Z = (1, 0, z_2\varepsilon)$ be the point on the line OU (neighbor to U). Then*:

(i) $A + B = ((a_1 + b_1) + (a_2 + b_2)\varepsilon, 0, 1)$.

(ii) $A + Z = (1, 0, z_2\varepsilon)$.

(iii) $A \cdot B = (a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon, 0, 1)$.

(iv) $A \cdot Z = (1, 0, (z_2a_1^{-1})\varepsilon)$ *where* $A \nsim O$.

(v) $Z \cdot A = (1, 0, (a_1^{-1}z_2)\varepsilon)$ *where* $A \nsim O$.

Proof Proof can be done following Definition 9 by using simple calculations.

(i)

$$\begin{aligned} A + B &= \bigl((AV \wedge OS)U \wedge BS\bigr)V \wedge OU \\ &= \bigl[1, 0, (a_1 + b_1) + (a_2 + b_2)\varepsilon\bigr] \wedge [0, 1, 0] \\ &= \bigl((a_1 + b_1) + (a_2 + b_2)\varepsilon, 0, 1\bigr). \end{aligned}$$

(ii)

$$\begin{aligned} A + Z &= \bigl((AV \wedge OS)U \wedge ZS\bigr)V \wedge OU \\ &= [z_2\varepsilon, 0, 1] \wedge [0, 1, 0] \\ &= (1, 0, z_2\varepsilon) = Z. \end{aligned}$$

(iii) If $B \nsim O$, then the inverse of b_1 exists and therefore

$$\begin{aligned} A \cdot B &= \big((BV \wedge 1S)O \wedge AS\big)V \wedge OU \\ &= \big[1, 0, a_1b_1 + \big(\big((a_1b_1)\big(b_1^{-1}b_2b_1^{-1}\big)\big)b_1 + a_2b_1\big)\varepsilon\big] \wedge [0,1,0] \\ &= \big(a_1b_1 + \big(\big((a_1b_1)\big(b_1^{-1}b_2b_1^{-1}\big)\big)b_1 + a_2b_1\big)\varepsilon, 0, 1\big) \end{aligned}$$

is obtained. Then using the associative and inversive property in $\mathbf{Q}$, we find $A \cdot B = (a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon, 0, 1)$.

If $B \sim O$, then

$$\begin{aligned} A \cdot B &= \big((BV \wedge 1S)O \wedge AS\big)V \wedge OU \\ &= [1, 0, a_1b_2\varepsilon,] \wedge [0,1,0] \\ &= (a_1b_2\varepsilon, 0, 1). \end{aligned}$$

(iv)

$$\begin{aligned} A \cdot Z &= \big((ZV \wedge 1S)O \wedge AS\big)V \wedge OU \\ &= \big[(z_2\varepsilon)(a_1 + a_2\varepsilon)^{-1}, 0, 1\big] \wedge [0,1,0] \\ &= \big(1, 0, (z_2\varepsilon)(a_1 + a_2\varepsilon)^{-1}\big) \\ &= \big(1, 0, \big(z_2a_1^{-1}\big)\varepsilon\big) \end{aligned}$$

is obtained.

(v)

$$\begin{aligned} Z \cdot A &= \big((AV \wedge 1S)O \wedge ZS\big)V \wedge OU \\ &= \big[\big(a_1^{-1}z_2\big)\varepsilon, 0, 1\big] \wedge [0,1,0] \\ &= \big(1, 0, \big(a_1^{-1}z_2\big)\varepsilon\big) \end{aligned}$$

is obtained. □

Theorem 11 *The properties given in Theorem* 10 *are independent of the choice of the point* S *given in Definition* 9, *where* S *is a point on* UV *and* $S \nsim V$, $S \nsim U$.

Proof If S' is an arbitrary point on UV non-neighbor to V, then there exist $s_1, s_2 \in \mathbf{Q}$ such that $S' = (1, s_1 + s_2\varepsilon, 0)$. We must show that the properties given in Theorem 10 hold when we replace S by S'.

(i)

$$\begin{aligned} A + B &= \big((AV \wedge OS')U \wedge BS'\big)V \wedge OU \\ &= \big[1, 0, (a_1 + b_1) + (a_2 + b_2)\varepsilon\big] \wedge [0,1,0] \\ &= \big((a_1 + b_1) + (a_2 + b_2)\varepsilon, 0, 1\big) \end{aligned}$$

is obtained.

(ii)

$$\begin{aligned} A + Z &= \big((AV \wedge OS')U \wedge ZS'\big)V \wedge OU \\ &= [z_2\varepsilon, 0, 1] \wedge [0, 1, 0] \\ &= (1, 0, z_2\varepsilon) = Z \end{aligned}$$

is obtained.

(iii) If $B \nsim O$, then b_1^{-1} exists and therefore

$$\begin{aligned} A \cdot B &= \big((BV \wedge 1S')O \wedge AS'\big)V \wedge OU \\ &= \big[1, 0, a_1b_1 + \big(\big((a_1b_1)\big(b_1^{-1}b_2b_1^{-1}\big)\big)b_1 + a_2b_1\big)\varepsilon\big] \wedge [0, 1, 0] \\ &= \big(a_1b_1 + \big(\big((a_1b_1)\big(b_1^{-1}b_2b_1^{-1}\big)\big)b_1 + a_2b_1\big)\varepsilon, 0, 1\big) \end{aligned}$$

is obtained. Then using the associative and inversive property in $\mathbf{Q}$, we find $A \cdot B = (a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon, 0, 1)$.

If $B \sim O$, then

$$A \cdot B = \big((BV \wedge 1S')O \wedge AS'\big)V \wedge OU = [1, 0, a_1b_2\varepsilon,] \wedge [0, 1, 0] = (a_1b_2\varepsilon, 0, 1).$$

(iv)

$$\begin{aligned} A \cdot Z &= \big((ZV \wedge 1S')O \wedge AS'\big)V \wedge OU \\ &= \big[(z_2\varepsilon)(a_1 + a_2\varepsilon)^{-1}, 0, 1\big] \wedge [0, 1, 0] \\ &= (1, 0, (z_2\varepsilon)(a_1 + a_2\varepsilon)^{-1} \\ &= \big(1, 0, \big(z_2a_1^{-1}\big)\varepsilon\big). \end{aligned}$$

(v)

$$\begin{aligned} Z \cdot A &= \big((AV \wedge 1S')O \wedge ZS'\big)V \wedge OU \\ &= \big[\big(a_1^{-1}z_2\big)\varepsilon, 0, 1\big] \wedge [0, 1, 0] = \big(1, 0, \big(a_1^{-1}z_2\big)\varepsilon\big). \end{aligned}$$

□

Theorem 12 *Let A and B be two non-neighbor points on OU and $A^* \sim A$, $B^* \sim B$, then $A + B \sim A^* + B^*$, $A \cdot B \sim A^* \cdot B^*$.*

Proof Let $A = (a_1 + a_2\varepsilon, 0, 1)$, $A^* = (a_1^* + a_2^*\varepsilon, 0, 1)$, $B = (b_1 + b_2\varepsilon, 0, 1)$ and $B^* = (b_1^* + b_2^*\varepsilon, 0, 1)$. We obtain $A + B = ((a_1 + b_1) + (a_2 + b_2)\varepsilon, 0, 1)$ and $A^* + B^* = ((a_1^* + b_1^*) + (a_2^* + b_2^*)\varepsilon, 0, 1)$. Then we have

$$\begin{aligned} A^* \sim A, \qquad B^* \sim B \quad &\Leftrightarrow \quad a_1^* = a_1, \qquad b_1^* = b_1 \\ &\Leftrightarrow \quad a_1^* + b_1^* = a_1 + b_1 \\ &\Leftrightarrow \quad A^* + B^* \sim A + B. \end{aligned}$$

Since $A \cdot B = (a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon, 0, 1)$ and $A^* \cdot B^* = (a_1^*b_1^* + (a_1^*b_2^* + a_2^*b_1^*)\varepsilon, 0, 1)$, we get

$$\begin{aligned} A^* \sim A, \qquad B^* \sim B \quad &\Leftrightarrow \quad a_1^* = a_1, \qquad b_1^* = b_1 \\ &\Leftrightarrow \quad a_1^* \cdot b_1^* = a_1 \cdot b_1 \\ &\Leftrightarrow \quad A^* \cdot B^* \sim A \cdot B. \end{aligned}$$

If $Z = (1, 0, z_2\varepsilon)$ and $Z^* = (1, 0, z_2^*\varepsilon)$, then $A + Z = Z$ and $A^* + Z^* = Z^*$ and it is trivial that

$$Z \sim Z^* \quad \Leftrightarrow \quad A + Z \sim A^* + Z^*.$$

Similarly, we have

$$A \cdot Z = \bigl(1, 0, (z_2a_1^{-1})\varepsilon\bigr) \sim \bigl(1, 0, (z_2^*a_1^{*-1})\varepsilon\bigr) = A^* \cdot Z^*$$

and

$$Z \cdot A = \bigl(1, 0, (a_1^{-1}z_2)\varepsilon\bigr) \sim \bigl(1, 0, (a_1^{*-1}z_2^*)\varepsilon\bigr) = Z^* \cdot A^*. \qquad \square$$

Corollary 13 *The following statements are valid where the points* A, B, Z, O *are defined as in Theorem* 10 *and* Y *is a point neighbor to* $(0, 0, 1)$ (*i.e.*, $Y \in \{(y_2\varepsilon, 0, 1) \mid y_2 \in \mathbf{Q}\}$):

(i) $A + B = B + A$ *and* $A + Z = Z + A$.
(ii) $A + O = A$ *and* $O + Z = Z$.
(iii) $A + Y \sim A$.
(iv) $A \cdot B \neq B \cdot A$.
(v) $O \cdot A = A \cdot O = O$.
(vi) $1 \cdot A = A = A \cdot 1$ *and* $1 \cdot Z = Z = Z \cdot 1$.
(vii) $A \cdot Y \sim Y$ *and* $Y \cdot A \sim Y$.

Proof (i) Since

$$\begin{aligned} \bigl((AV \wedge OS)U \wedge BS\bigr)V &= \bigl[1, 0, (a_1 + b_1) + (a_2 + b_2)\varepsilon\bigr] \\ &= \bigl[1, 0, (b_1 + a_1) + (b_2 + a_2)\varepsilon\bigr] \\ &= \bigl((BV \wedge OS)U \wedge AS\bigr)V, \end{aligned}$$

we obtain $A + B = B + A$. And since

$$\bigl((AV \wedge OS)U \wedge ZS\bigr)V = [z_2\varepsilon, 0, 1] = \bigl((ZV \wedge OS)U \wedge AS\bigr)V,$$

we obtain $A + Z = Z + A$.

(ii)

$$A + O = \bigl((AV \vee OS)U \wedge OS\bigr)V \wedge OU = [1, 0, a_1 + a_2\varepsilon] \wedge [0, 1, 0] = A$$

is obtained.

Similarly, we get

$$\begin{aligned} O + Z &= \big((OV \wedge OS)U \wedge ZS\big)V \wedge OU \\ &= [z_2\varepsilon, 0, 1] \wedge [0,1,0] = Z. \end{aligned}$$

(iii) It is trivial from Theorem 12.

(iv) Let $A = (i, 0, 1)$ and $B = (j, 0, 1)$. Since $A \cdot B = (k, 0, 1)$ and $B \cdot A = (-k, 0, 1)$, we have $A \cdot B \neq B \cdot A$.

(v) By simple calculations,

$$\begin{aligned} O \cdot A &= \big((AV \wedge 1S)O \wedge OS\big)V \wedge OU \\ &= [1,0,0] \wedge [0,1,0] \\ &= \big((OV \wedge 1S)O \wedge AS\big)V \wedge OU \\ &= A \cdot O \end{aligned}$$

is obtained.

Also, since $[1,0,0] \wedge [0,1,0] = (0,0,1) = O$, we have $O \cdot A = A \cdot O = O$.

(vi) Since

$$\begin{aligned} A \cdot 1 &= \big((1V \wedge 1S)O \wedge AS\big)V \wedge OU \\ &= [1, 0, a_1 + a_2\varepsilon] \wedge [0,1,0] \\ &= \big((AV \wedge 1S)O \wedge 1S\big)V \wedge OU \\ &= 1 \cdot A \end{aligned}$$

and $[1, 0, a_1 + a_2\varepsilon] \wedge [0,1,0] = (a_1 + a_2\varepsilon, 0, 1) = A$, we get $A \cdot 1 = 1 \cdot A = A$.

Similarly,

$$\begin{aligned} 1 \cdot Z &= \big((ZV \wedge 1S)O \wedge 1S\big)V \wedge OU \\ &= [z_2\varepsilon, 0, 1] \wedge [0,1,0] = Z \cdot 1, \end{aligned}$$

and hence we get $[z_2\varepsilon, 0, 1] \wedge [0,1,0] = (1, 0, z_2\varepsilon) = Z$.

(vii) Since $Y = (y_2\varepsilon, 0, 1)$,

$$A \cdot Y = \big((YV \wedge 1S)O \wedge AS\big)V \wedge OU = [1, 0, a_1 y_2 \varepsilon,] \wedge [0,1,0] = (a_1 y_2 \varepsilon, 0, 1)$$

and

$$Y \cdot A = \big((AV \wedge 1S)O \wedge YS\big)V \wedge OU = \big[1, 0, (y_2\varepsilon)(a_1 + a_2\varepsilon)\big] \wedge [0,1,0] = (y_2 a_1 \varepsilon, 0, 1),$$

we have $A \cdot Y \sim Y$ and $Y \cdot A \sim Y$. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors have contributed equally to this paper. Both authors read and approved the final manuscript.

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