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The tangent spherical image and ccr-curve of a time-like curve in $\mathbb{L}^3$

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Abstract

In this work, we define the tangent spherical image of a unit speed time-like curve lying on the pseudohyperbolic space $H_0^2(r)$ in $\mathbb{L}^3$. In addition, we calculate a ccr-curve of this curve in $\mathbb{L}^3$. Besides, we determine a relation between harmonic curvature and a ccr-curve in $\mathbb{L}^3$, and so we obtain some new results.

Introduction

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be two non-zero vectors in the three-dimensional Lorentz-Minkowski space $\mathbb{R}_1^3$. We denoted $\mathbb{R}_1^3$ shortly by $\mathbb{L}^3$. For $X, Y \in \mathbb{L}^3$,

$$\langle X, Y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

is called a *Lorentzian inner product*. The couple $\{\mathbb{R}_1^3, \langle\ ,\ \rangle\}$ is called a *Lorentzian space* and denoted by $\mathbb{L}^3$. Then the vector X of $\mathbb{L}^3$ is called

(i) time-like if $\langle X, X \rangle < 0$,
(ii) space-like if $\langle X, X \rangle > 0$ or $X = 0$,
(iii) a null (or light-like) vector if $\langle X, X \rangle = 0$, $X \neq 0$.

The norm of a vector X is given by $\|X\| = \sqrt{|\langle X, X \rangle|}$. Therefore, X is a unit vector if $\langle X, X \rangle = \pm 1$. Next, vectors X, Y in $\mathbb{L}^3$ are said to be orthogonal if $\langle X, Y \rangle = 0$. The velocity of a curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Space-like or time-like $\alpha(s)$ is said to be parametrized by an arclength function s if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$ [1]. For any $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}_1^3$, the pseudo-vector product of a X and Y is defined as follows:

$$X \wedge Y = \bigl(-(x_2 y_3 - x_3 y_2), x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\bigr)$$

[2].

1 Basic concepts

Definition 1.1 An arbitrary curve $\alpha : I \longrightarrow \mathbb{L}^3$ in the space $\mathbb{L}^3$ can locally be space-like, time-like or a null curve if, respectively, all of its velocity vectors $\alpha'(s)$ are space-like, time-like or null [3].

Definition 1.2 Let $\alpha \subset \mathbb{L}^3$ be a given time-like curve. If the Frenet vector $\{V_1(s), V_2(s), V_3(s)\}$ which corresponds to $s \in I$ is defined as

$$k_i : I \longrightarrow \mathbb{R}, \quad k_i(s) = \bigl\langle V_i'(s), V_{i+1}(s) \bigr\rangle,$$

then the function k_i is called an ith curvature function of the time-like curve α, and the real $k_i(s)$ is also called an ith curvature at the point $\alpha(s)$ [4].

Definition 1.3 Let $\alpha : I \longrightarrow \mathbb{L}^3$ be a unit speed non-null curve in $\mathbb{L}^3$. The curve α is called a Frenet curve of osculating order d ($d \leq 3$) if its 3rd order derivatives $\alpha'(s)$, $\alpha''(s)$, $\alpha'''(s)$ are linearly independent and $\alpha'(s)$, $\alpha''(s)$, $\alpha'''(s)$, $\alpha^{iv}(s)$ are no longer linearly independent for all $s \in I$. For each Frenet curve of order 3, one can associate an orthonormal 3-frame $\{V_1(s), V_2(s), V_3(s)\}$ along α (such that $\alpha'(s) = V_1$) called the Frenet frame and $k_1, k_2 : I \longrightarrow \mathbb{R}$ called the Frenet curvatures, such that the Frenet formulas are defined in the usual way:

$$V_1' = \nabla_{v_1}\alpha' = \varepsilon_2 k_1 V_2,$$
$$V_2' = \nabla_{v_1} V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3,$$
$$V_3' = \nabla_{v_1} V_3 = -\varepsilon_2 k_2 V_2,$$

where ∇ is the Levi-Civita connection of $\mathbb{L}^3$.

Definition 1.4 A non-null curve $\alpha : I \longrightarrow \mathbb{L}^3$ is called a W-curve (or helix) of rank 3, if α is a Frenet curve of osculating order 3 and the Frenet curvatures k_i, $1 \leq i \leq 2$, are non-zero constants.

2 Harmonic curvatures and constant curvature ratios in $\mathbb{L}^3$

Definition 2.1 Let α be a non-null curve of osculating order 3. The harmonic functions

$$H_j : I \longrightarrow \mathbb{R}, \quad 0 \leq j \leq 1,$$

defined by

$$\begin{cases} H_0 = 0, \\ H_1 = \frac{k_1}{k_2} \end{cases}$$

are called harmonic curvatures of α, where k_1, k_2 are Frenet curvatures of α which are not necessarily constant.

Definition 2.2 Let α be a time-like curve in $\mathbb{L}^3$ with $\alpha'(s) = V_1$. $X \epsilon \chi(\mathbb{L}^3)$ being a constant unit vector field, if

$$\langle V_1, X\rangle = \cosh\varphi \quad \text{(constant)},$$

then α is called a general helix (inclined curves) in $\mathbb{L}^3$, φ is called a slope angle and the space $Sp\{X\}$ is called a slope axis [5].

Definition 2.3 Let α be a non-null of osculating order 3. Then α is called a general helix of rank 1 if

$$H_1^2 = c,$$

holds, where $c \neq 0$ is a real constant.

We have the following results.

Corollary 2.1

(i) *If* $H_1 = 0$, *then* α *is a straight line.*

(ii) *If* H_1 *is constant, then* α *is a general helix of* rank 1.

Proof By the use of the above definition, we obtain the proof. □

Proposition 2.1 *Let* α *be a curve in* $\mathbb{L}^3$ *of osculating order* 3. *Then*

$$V_1' = \varepsilon_2 k_2 H_1 V_2,$$
$$V_2' = -\varepsilon_1 k_2 H_1 V_1 + \varepsilon_3 \frac{k_1}{H_1} V_3,$$
$$V_3' = -\varepsilon_2 \frac{k_1}{H_1} V_2,$$

where H_1 *is harmonic curvature of* α.

Proof By using the Frenet formulas and the definitions of harmonic curvatures, we get the result. □

Now, we will give the relation between harmonic curvature and a ccr-curve in $\mathbb{L}^3$.

Definition 2.4 A curve $\alpha : I \longrightarrow \mathbb{L}^3$ is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients $\varepsilon_i(\frac{k_{i+1}}{k_i})$ are constant, where $\varepsilon_i = \langle V_i, V_i\rangle = \pm 1$.

Corollary 2.2 *For* $i = 1$, *the ccr-curve is* $\frac{\varepsilon_1}{H_1}$.

Proof The proof can be easily seen by using the definitions of harmonic curvature and ccr-curve. □

Corollary 2.3 *Let* $\alpha : I \longrightarrow \mathbb{L}^3$ *be a ccr-curve. If* $\frac{\varepsilon_1}{H_1} = c$, c *is a constant, then* $(\frac{\varepsilon_1}{H_1})' = 0$.

Proof The proof is obvious. □

3 Tangent spherical image

Definition 3.1 [1] Let $n \geq 2$ and $0 \leq \nu \leq n$. Then the pseudohyperbolic space of radius $r > 0$ in $\mathbb{R}_1^3$ is the hyperquadric

$$H_0^2(r) = \{p \in \mathbb{R}_1^3 : \langle p, p\rangle = -r^2\}$$

with dimension 2 and index 0.

Definition 3.2 Let $\alpha = \alpha(s)$ be a unit speed time-like curve in $\mathbb{L}^3$. If we translate the tangent vector to the center 0 of the pseudohyperbolic space $H_0^2(r)$, we obtain a curve $\delta = \delta(s_\delta)$. This curve is called the tangent spherical image of a curve α in $\mathbb{L}^3$.

Theorem 3.1 [6]

(i) *Let $\alpha = \alpha(s)$ be a unit speed time-like curve and $\delta = \delta(s_\delta)$ be its tangent spherical image. Then $\delta = \delta(s_\delta)$ is a space-like curve.*

(ii) *Let $\alpha = \alpha(s)$ be a unit speed time-like curve and $\delta = \delta(s_\delta)$ be its tangent spherical image. If α is a ccr-curve or a helix (i.e. W-curve), then δ is also a helix.*

Proof From [6] it is easy to see the proof of the theorem. □

4 An example

Example 4.1 Let us consider the following curve in the space $\mathbb{L}^3$:

$$\alpha(s) = (\sqrt{2}s, \cos s, \sin s),$$
$$V_1(s) = \alpha'(s) = (\sqrt{2}, -\sin s, \cos s),$$

where $\langle \alpha'(s), \alpha'(s)\rangle = -1$, which shows $\alpha(s)$ is a unit speed time-like curve. Thus $\|\alpha'(s)\| = 1$. We express the following differentiations:

$$\alpha''(s) = (0, -\cos s, -\sin s),$$
$$\Rightarrow \quad \alpha'''(s) = (0, \sin s, -\cos s)$$

and

$$V_2(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|} = \alpha''(s).$$

So, we have the first curvature as

$$k_1(s) = \langle V_1'(s), V_2(s)\rangle = 1 = \text{constant}.$$

Moreover, we can write the third Frenet vector of the curve as follows:

$$V_3(s) = V_1(s) \Lambda V_2(s) = (-1, \sqrt{2}\sin s, -\sqrt{2}\cos s).$$

Finally, we have the second curvature of $\alpha(s)$ as

$$k_2(s) = \langle V_2'(s), V_3(s)\rangle = \sqrt{2} = \text{constant}.$$

Now, we will calculate a ccr-curve of $\alpha(s)$ in $\mathbb{L}^3$. If the vector V_1 is time-like, then $\varepsilon_1 = -1$,

$$\varepsilon_1 \frac{k_2}{k_1} = -\sqrt{2} = \text{constant}.$$

Thus $\alpha(s)$ is a ccr-curve in $\mathbb{L}^3$.

Competing interests

The author declares that they have no competing interests.

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