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A Class of 3-dimensional Contact Metric Manifolds

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Abstract. We classify the contact metric 3-manifolds such that $\|grad\lambda\| = 1$ and $\nabla_\xi \tau = 2a\tau\phi$.

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1. Introduction

Let $M(\phi, \xi, \eta, g)$ be a contact metric maniold. It is well known that the tensors $\tau = \mathcal{L}_\xi g$ and $\nabla_\xi \tau$ play a fundamental role in the study of the geometry of $M(\phi, \xi, \eta, g)$ (here, $\mathcal{L}_\xi$ is the Lie derivative in the direction of ξ). A contact metric 3-manifold is said to be 3-τ-a if it satisfies

$$\nabla_\xi \tau = 2a\tau\phi, \tag{1.1}$$

where a is an arbitrary smooth function on M [10].

The classification of conformally flat contact metric manifolds is an interesting problem which has been investigated by many researchers. At one hand, in many cases conformally flat contact metric manifolds must have constant sectional curvature ([8], [13]). On the other hand, Blair [3, pp.108] constructed examples of non-compact conformally flat contact metric 3-manifolds with non-constant sectional curvature. In [6], Calvaruso proved that a conformally flat contact metric 3-manifold satisfying (1.1) has constant sectional curvature 0 or 1 and showed that Blair's examples satisfy the $\nabla_\xi \tau = 2a\tau\phi$, where a smooth function with $\xi(a) = 0$. Gouli-Andreou et.al [10] found a new class of conformaly flat 3-τ-a manifold and constructed compact examples of conformally flat contact metric 3-manifolds.

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As a generalization of the Sasakian manifold, Blair et al. [3] introduced the notion of a contact metric manifold called a (κ, μ)-contact metric manifold satisfying the condition

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{1.2}$$

for all vector fields X, Y on M, where κ and μ are constants on M. Recently (κ, μ)-contact metric manifolds have been studied by various authors ([5], [9]). (κ, μ)-contact metric manifolds include Sasakian manifolds ($\kappa = 1$ and $h = 0$), and also many examples of non-Sasakian (κ, μ)-contact metric manifolds have been provided. Koufogiorgos and Tsichlias [17] generalized the notion of a (κ, μ)-contact metric manifold by regarding the constants κ and μ in (1.2) to be smooth functions on M, called a generalized (κ, μ)-contact metric manifold. It is proved in [17] that if $\dim M > 3$ then κ, μ were necessarely constant. Moreover, they gave the examples satisfying (1.2) with κ, μ non constant smooth functions for dimension 3 in [17]. The local classification of 3-dimensional generalized (κ, μ)-contact metric manifolds, satisfying the condition $\|grad\kappa\|$ =constant($\neq 0$) was obtained in [18]. In [14] Koufogiorgos et al. proved the existence of a new class of contact metric manifolds: the so called (κ, μ, v)-contact metric manifolds. Such a manifold M is defined through the condition

$$\begin{aligned} R(X,Y)\xi &= \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ &\quad +\nu(\eta(Y)\phi hX - \eta(X)\phi hY) \end{aligned} \tag{1.3}$$

for all vector fields X, Y on M and κ, μ, ν are smooth functions on M. Furthermore, it is shown in [14] that if dim $M > 3$, then κ, μ are constants and ν is the zero function. They also proved that the condition (1.3) is invariant under the D-homothetic deformations, and further that, if $\dim M = 3$, then the condition (1.3) is equivalent to the following condition

$$Q = \left(\frac{\rho}{2} - \kappa\right) I + \left(-\frac{\rho}{2} + 3\kappa\right) \eta \otimes \xi + \mu h + \nu \phi h \tag{1.4}$$

holding on an open and dense subset of M, where ρ is the scalar curvature of M. From (1.4) it can be easily obtained that the characteristic vector field ξ is an eigenvector of the Ricci operator Q. Koufogiorgos et al. [15] gave a classification of 3-τ-a for (κ, μ, v)-contact metric manifolds.

On a compact orientable m-dimensional Riemannian manifold (M, g), a unit vector field V is said to be harmonic if it is a critical point for the energy functional, $E(V) = \frac{m}{2} vol(M,g) + \frac{1}{2} \int_M \| \nabla V \|^2 \, dv$, on the space of all unit vector fields. A $(2n+1)$-dimensional contact metric manifold $M(\phi, \xi, \eta, g)$ whose characteristic vector field ξ is a harmonic vector field is called a H-contact metric manifold. Perrone [20] proved that $M(\phi, \xi, \eta, g)$ is H-contact metric manifold if and only if ξ is an eigenvector of the Ricci operator Q. Perrone [19] also gave a geometric interpretation of generalized (κ, μ)-contact metric manifolds in terms of harmonic maps. In particular, he showed that a contact metric 3-manifold M is a generalized (κ, μ)-contact metric manifold on an everywhere dense open subset of M if and only if its characteristic vector field ξ determines a harmonic map. Then Koufogiorgos et al. [14] proved that a contact metric 3-manifold M is an H-contact metric manifold if and only if it is a (κ, μ, ν)-contact metric manifold on an everywhere dense and open subset of M. In the same paper, they also gave examples of 3-dimensional (κ, μ, v)-contact metric manifolds which are not generalized (κ, μ)-contact metric manifolds. Koufogiorgos and Stamatiou [16] also showed that a 3-dimensional contact metric manifold M with $R(X, Y)\xi = 0$, for any $X, Y \in \ker\eta$, is a (κ, μ, v)-contact metric manifold on an open and dense subset of M.

A contact metric manifold M is called weakly locally ϕ-symmetric if it satisfies the curvature condition $g((\nabla_X R)(Y, Z)V, W) = 0$ for all vector fields X, Y, Z, V and W orthogonal to the characteristic vector field ξ (as in the Sasakian case) [4]. A contact metric manifold M is called a strongly locally ϕ-symmetric contact metric manifold if the characteristic reflections are local isometries ([8], [16]). Calvaruso et al. [8] proved that, in dimension three, M is strongly locally ϕ-symmetric if and only if M is a generalized (κ, μ)-contact metric manifold.

In this paper, we obtain a full local classification of 3-dimensional (non-Sasakian) contact metric manifolds satisfying $\|grad\lambda\| = 1$ ($\lambda, -\lambda$ being the nonvanishing eigenvalues of tensor h) and $\nabla_\xi\tau = 2a\tau\phi$. The paper is organized in the following way. The Section 2 contains the presentation of some basic notions about contact manifolds and (κ, μ)-contact metric manifolds, (κ, μ, v)-contact metric manifolds. In section 3 we give some properties of contact metric 3-manifold. In section 4 we give Main Theorem and generalize the corollaries of [15] and [18] .We also give two examples which satisfy the conditions of this manifold. In section 5 we give several properties and some applications about contact metric 3-manifold with $\|grad\ \lambda\| = 1$ and $\nabla_\xi\tau = 2a\tau\phi$.

2. Preliminaries

A differentiable manifold M of dimension $2n+1$ is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that then there exists a unique vector field ξ (called the Reeb vector field) such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. It is well known that there also exists a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\phi(\xi) = 0, \quad \phi^2 = -I + \eta\otimes\xi, \quad \eta\circ\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector field X and Y on M. The structure (ϕ, ξ, η, g) can be chosen so that $d\eta(X, Y) = g(X, \phi Y)$. The manifold M together with the structure tensors (ϕ, ξ, η, g) is called a contact metric manifold structure and is denoted by $M(\phi, \xi, \eta, g)$. Define an operator h by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes Lie differentiation. The tensor field h vanishes identically if and only if the vector field ξ is Killing and in this case the contact metric manifold is said to be

K-contact. It is well known that h and ϕh are symmetric operators, h anti-commutes with ϕ

$$\phi h + h\phi = 0,\ h\xi = 0,\ \eta \circ h = 0,\ trh = \ tr\phi h = 0, \tag{2.3}$$

where trh denotes the trace of h. Since h anti-commutes with ϕ, if X is an eigenvector of h corresponding to the eigenvalue λ then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. Moreover, for any contact manifold M, the following is satisfied

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.4}$$

where ∇ is the Riemannian connection of g.

On a contact metric manifold M^{2n+1} we have the formulas

$$\begin{aligned}
(\nabla_\xi h) &= \phi(I - h^2 - l), & (2.5)\\
l - \phi l \phi &= -2(h^2 + \phi^2), & (2.6)\\
Trl &= g(Q\xi, \xi) = 2n - Trh^2, & (2.7)\\
\tau &= 2g(\phi\cdot, h\cdot), & (2.8)\\
\nabla_\xi \tau &= 2g(\phi\cdot, \nabla_\xi h\cdot), & (2.9)\\
\|\tau\|^2 &= 4\mathrm{tr}h^2, & (2.10)
\end{aligned}$$

where $l = R(X,\xi)\xi$, Q is Ricci operator of M.

A contact metric manifold satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$ ([2]).

If a contact metric manifold M is normal (i.e., $N_\phi + 2d\eta \otimes \xi = 0$, where N_ϕ denotes the *Nijenhuis tensor* formed with ϕ), then M is called a *Sasakian manifold*. Equivalently, a contact metric manifold is Sasakian if and only if $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$ or $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$ ([1]).

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian manifold consider

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2.11}$$

for smooth functions κ and μ on M. If κ, μ is constant M is called (κ,μ)-contact metric manifold. Otherwise M is generalized (κ,μ)-contact metric manifold. This kind of manifolds were introduced and studied by Blair, Koufogiorgos and Papantoniou in [3]. Since then, they have been intensively studied see in particular [5] and [14].

Let $M(\phi,\xi,\eta,g)$ be a contact metric manifold. A D-homothetic transformation [21] is the transformation

$$\bar{\eta} = \alpha\eta, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta \tag{2.12}$$

at the structure tensors, where α is a positive constant. It is well known (see [21]) that $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also a contact metric manifold. When two contact structures (ϕ,ξ,η,g) and $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ are related by (2.12), we will say that they are D-homothetic.

We can easily show that $\bar{h} = \frac{1}{\alpha}h$ so $\bar{\lambda} = \frac{1}{\alpha}\lambda$. Using the relations above we finally obtain that

$$\begin{aligned}\bar{R}(X,Y)\bar{\xi} = &\frac{\kappa+\alpha^2-1}{\alpha^2}(\bar{\eta}(Y)X - \bar{\eta}(X)Y)\\ &+ \frac{\mu+2(\alpha-1)}{\alpha}(\bar{\eta}(Y)\bar{h}X - \bar{\eta}(X)\bar{h}Y)\end{aligned}$$

for all vector fields X and Y on M. Thus $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a $(\bar{\kappa},\ \bar{\mu})$-contact metric manifold with

$$\bar{\kappa} = \frac{\kappa+\alpha^2-1}{\alpha^2}, \quad \bar{\mu} = \frac{\mu+2(\alpha-1)}{\alpha}.$$

It is well known(see, for example, [2]) that every 3-dimensional contact metric manifold satisfies the integrability condition

$$(\nabla_X\phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX).$$

Now we will give examples of generalized (κ,μ)-contact metric manifold and generalized (κ,μ,ν)-contact metric manifold.

Example. [15, 17] We consider the 3-dimensional manifold $M = \{(x_1,x_2,x_3) \in R^3 \mid x_3 \neq 0\}$, where (x_1,x_2,x_3) are the standard coordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial x_1},\ e_2 = -2x_2x_3\frac{\partial}{\partial x_1} + \frac{2x_1}{x_3^3}\frac{\partial}{\partial x_2} - \frac{1}{x_3^2}\frac{\partial}{\partial x_3},\ e_3 = \frac{1}{x_3}\frac{\partial}{\partial x_2}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by $g(e_i,e_j) = \delta_{ij},\ i,j = 1,2,3$ and η the dual 1-form to the vector field e_1.We define the tensor ϕ of type $(1,1)$ by $\phi e_1 = 0, \phi e_2 = e_3, \phi e_3 = -\ e_2$. Following [17], we have that $M(\eta, e_1, \phi, g)$ is a generalized (κ,μ)-contact metric manifold with $\kappa = \frac{x_3^4-1}{x_3^4}$, $\mu = 2\left(1 - \frac{1}{x_3^2}\right)$. By a straightforward calculation, one can deduce that M satisfies

$$\nabla_\xi\tau = 2\left(1 - \frac{1}{x_3^2}\right)\tau\phi.$$

In [14] Koufogiorgos et al. proved the existence of a new class of contact metric manifolds which is called (κ,μ,υ)-contact metric manifold. This means that curvature tensor R satisfies the condition

$$\begin{aligned}R(X,Y)\xi &= \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)\\ &\quad +\upsilon(\eta(Y)\phi hX - \eta(X)\phi hY)\end{aligned}$$

for any vector fields X, Y and κ,μ,υ are smooth functions.

Furthermore, it is shown in [14] that if $dimM > 3$, then κ,μ are constants and υ is the zero function.

Example. [14] Let M be 3-dimensional contact metric manifold such that

$$M = \left\{(x,y,z) \in R^3 \mid x > 0,\ y > 0,\ z > 0\right\},$$

where (x, y, z) are the cartesian coordinates in R^3. We define three vector fields on M as

$$e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\quad e_3 = -\frac{4}{z}e^G\ G_y\frac{\partial}{\partial x} + \beta\ \frac{\partial}{\partial y} + e^{G/2}\ \ \frac{\partial}{\partial z},$$

where $G = G(y, z) < 0$ for all (y, z) is a solution of the partial differential equation

$$2G_{yy} + G_y^2 = -ze^{-G}$$

and the function $\beta = \beta(x, y, z)$ solves the system of partial differential equations

$$\begin{aligned}\beta_x &= \frac{4}{zx^2}e^G,\\ \beta_y &= \frac{1}{2z}e^{G/2} - \frac{G_z e^{G/2}}{2} - \frac{4e^G G_y}{xz}.\end{aligned}$$

Setting $\kappa = 1 - (4e^{2G)}/(z^2x^4)$, $\mu = 2(1 + (2e^G)/(zx^2))$ and $\nu = -2/x$. By direct calculation, these relations yield

$$\begin{aligned}R(Z, W)\xi &= \kappa(\eta(W)Z - \eta(Z)W) + \mu(\eta(W)hZ - \eta(Z)hW) +\\ &\quad +\nu(\eta(W)\phi hZ - \eta(Z)\phi hW)\end{aligned}$$

for all vector fields Z, W on M, where κ, μ, ν are nonconstant smooth functions. Hence, it has been shown that M is a (generalized) (κ, μ, ν)-contact metric manifold.

3. Three dimensional contact metric manifolds

In this section, we will give some properties of contact metric 3-manifold.

Let $M(\phi, \xi, \eta, g)$ be a contact metric 3-manifold. Let

$$\begin{aligned}U &= \{p \in M \mid h(p) \neq 0\} \subset M,\\ U_0 &= \{p \in M \mid h(p) = 0\} \subset M.\end{aligned}$$

That h is a smooth function on M implies $U \cup U_0$ is an open and dense subset of M, so any property satisfied in $U_0 \cup U$ is also satisfied in M.

For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (this we call a ϕ-basis).

On U, we put $he = \lambda e$, $h\phi e = -\lambda\phi e$, where λ is a nonvanishing smooth function assumed to be positive.

Lemma 3.1. [12] *(see also* [8]*) On the open set U we have*

$$\nabla_\xi e = a\phi e,\ \nabla_e e = b\phi e,\ \nabla_{\phi e} e = -c\phi e + (\lambda - 1)\xi, \tag{3.1}$$

$$\nabla_\xi \phi e = -ae, \nabla_e \phi e = -be + (1 + \lambda)\xi, \nabla_{\phi e}\phi e = ce, \tag{3.2}$$

$$\nabla_\xi \xi = 0,\ \nabla_e \xi = -(1 + \lambda)\phi e,\ \nabla_{\phi e}\xi = (1 - \lambda)e, \tag{3.3}$$

$$\nabla_\xi h = -2ah\phi + (\xi \cdot \lambda)s, \tag{3.4}$$

where a is a smooth function,

$$b = \frac{1}{2\lambda}((\phi e \cdot \lambda) + A) \quad with \quad A = \eta(Qe) = S(\xi, e), \tag{3.5}$$

$$c = \frac{1}{2\lambda}((e \cdot \lambda) + B) \quad with \quad B = \eta(Q\phi e) = S(\xi, \phi e) \tag{3.6}$$

and s is the type $(1,1)$ tensor field defined by $s\xi = 0$, $se = e$ and $s\phi e = -\phi e$.

From Lemma 3.1 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$, we can prove that

$$[e, \phi e] = \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \tag{3.7}$$

$$[e, \xi] = \nabla_e \xi - \nabla_\xi e = -(a + \lambda + 1)\phi e, \tag{3.8}$$

$$[\phi e, \xi] = \nabla_{\phi e} \xi - \nabla_\xi \phi e = (a - \lambda + 1)e. \tag{3.9}$$

Definition 3.2. [10] Let M be a 3-dimensional contact metric manifold and $h = \lambda h^+ - \lambda h^-$ the spectral decomposition of h on U_1. If

$$\nabla_{h^- X} h^- X = \left[\xi, h^+ X\right] \tag{3.10}$$

for all vector fields X on M and all points of an open subset W of U_1 and $h = 0$ on the points of M which do not belong to W, then the manifold is said to be semi-K contact manifold.

Remark 3.3. [10] From relations (3.1)-(3.4) and (3.7)-(3.9) the condition (3.10) for $X = e$ leads to $[\xi, e] = 0$ while for $X = \phi e$ leads to $\nabla_{\phi e} \phi e = 0$. Hence on a semi-$K$ contact manifold we have $a + \lambda + 1 = c = 0$. If we apply the deformation $e \to \phi e$, $\phi e \to e$, $\xi \to -\xi$, $\lambda \to -\lambda$, $b \to c$, $c \to b$, then the contact structure remains the same. Hence the condition for a 3-dimensional contact metric manifold to be semi-K contact is equivalent to $a - \lambda + 1 = b = 0$. On the other hand, if on a 3-dimensional contact metric manifold the relation $\nabla_{h^+ X} h^+ X = [\xi, h^- X]$ holds, then applying relations (3.1)-(3.4) we have $a - \lambda + 1 = b = 0$.

4. Main Results

The local classification of 3-dimensional generalized (κ, μ)-contact metric manifolds, satisfying the condition $\|grad\kappa\|$ =constant($\neq 0$) was obtained in [18]. As a result, contact metric manifold with $\|grad\ \lambda\|_g = d \neq 0$ (cons.) is D_α- deformed in another contact metric manifold with $\left\|grad\ \bar{\lambda}\right\|_{\bar{g}} = d\alpha^{-\frac{3}{2}}$ and choosing $\alpha = d^{\frac{2}{3}}$, it is enough to study those contact metric manifold with $\|grad\ \lambda\| = 1$. If $d = 0$, then λ is constant. As a result, if $\lambda = 0$, then M is a Sasakian manifold.

Now we will give our main Theorem.

Theorem 4.1. *Let $M(\phi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold with $\|grad\ \lambda\| = 1$ and $\nabla_\xi \tau = 2a\tau\phi$. Then at any point $p \in M$ there exist a chart $(U, (x, y, z))$ such that $\lambda = g(z) \neq 0$ and $A = 0, B = F(y, z)$ or*

$A = F(y,z), B = 0$. In the first case ($A = 0, B = F(y,z)$), the following are valid,

$$\xi = \frac{\partial}{\partial x},\ \phi e = \frac{\partial}{\partial y}\ \text{and}\ e = k_1\frac{\partial}{\partial x} + k_2\frac{\partial}{\partial y} + k_3\frac{\partial}{\partial z},\quad k_3 \neq 0.$$

In the second case ($A = F(y,z), B = 0$), the following are valid,

$$\xi = \frac{\partial}{\partial x},\ e = \frac{\partial}{\partial y}\ \text{and}\ \phi e = k_1'\frac{\partial}{\partial x} + k_2'\frac{\partial}{\partial y} + k_3'\frac{\partial}{\partial z},\quad k_3' \neq 0,$$

where

$$k_1(x,y,z) = -2y + r(z),\quad k_1'(x,y,z) = 2y + r'(z),$$

$$k_2(x,y,z) = k_2'(x,y,z) = 2xg(z) - \frac{(H(y,z)+y)}{2g(z)} + \beta(z),$$

$$k_3(x,y,z) = k_3'(x,y,z) = t(z) + \delta,\qquad \frac{\partial H(y,z)}{\partial y} = F(y,z)$$

and r, r', β are smooth functions of z and δ is constant. Also, $g(z) = \int \frac{1}{k_3(z)}dz$.

Proof. By virtue of (2.9) and (2.10), it can be proved that the assumption $\nabla_\xi \tau = 2a\tau\phi$ is equivalent to $\xi \cdot \lambda = 0$. From the definition of gradient of a differentiable function we get

$$\begin{aligned} grad\lambda &= (e\cdot\lambda)e + (\phi e\cdot\lambda)\phi e + (\xi\cdot\lambda)\xi \\ &= (e\cdot\lambda)e + (\phi e\cdot\lambda)\phi e. \end{aligned} \tag{4.1}$$

Using (4.1) and $\|grad\ \lambda\| = 1$, we have

$$(e\cdot\lambda)^2 + (\phi e\cdot\lambda)^2 = 1. \tag{4.2}$$

Differentiating (4.2) with respect to ξ and using (3.8) and (3.9) we obtain

$$\begin{aligned} (\xi.e(\lambda)(e(\lambda) + (\xi\phi e(\lambda))(\phi e(\lambda) &= 0 \\ ([\xi,e](\lambda))\,e(\lambda) + ([\xi,\phi e](\lambda))\,(\phi e)\lambda &= 0 \\ \lambda e(\lambda)\phi e(\lambda) &= 0 \end{aligned}$$

and, since $\lambda \neq 0$,

$$e(\lambda)\phi e(\lambda) = 0. \tag{4.3}$$

To study this system we consider the open subsets of U

$$U' = \{p \in U \mid e(\lambda)(p) \neq 0\,\},\ \ U'' = \{p \in U \mid (\phi e)(\lambda)p \neq 0\},$$

where $U' \cup U''$ is open and dense in the closure of U. We distinguish two cases:

Case 1: Now we suppose that $p \in U'$. By virtue of (4.2), (4.3) we have $(\phi e)(\lambda) = 0, e(\lambda) = \mp 1$. Changing to the basis $(\xi, -e, -\phi e)$ if necessary, we can assume that $e(\lambda) = 1$. By the equation (3.9), we get

$$\begin{aligned} [\phi e,\xi](\lambda) &= (\phi e)(\xi(\lambda)) - \xi((\phi e)(\lambda)) \\ &= (a - \lambda + 1)e(\lambda). \end{aligned} \tag{4.4}$$

If we use the relations $e(\lambda)=1,(\phi e)(\lambda)=0$ and $\xi\cdot\lambda=0$ in the equation (4.4), one can easily obtain $a=\lambda-1$. Hence, the equations (3.7), (3.8), (3.9) and (3.5), (3.6) are reduced

$$[e,\phi e] = -be+c\phi e+2\xi, \tag{4.5}$$
$$[e,\xi] = -2\lambda\phi e, \tag{4.6}$$
$$[\phi e,\xi] = 0, \tag{4.7}$$
$$b=\frac{A}{2\lambda},\quad c=\frac{(B+1)}{2\lambda}, \tag{4.8}$$

respectively.

Since $[\phi e,\xi]=0$, the distribution which is spanned by ϕe and ξ is integrable and so for any $p\in U'$ there exist a chart $\{V,(x,y,z)\}$ at p, such that

$$\xi=\frac{\partial}{\partial x},\quad \phi e=\frac{\partial}{\partial y},\quad e=k_1\frac{\partial}{\partial x}+k_2\frac{\partial}{\partial y}+k_3\frac{\partial}{\partial z}, \tag{4.9}$$

where k_1,k_2,k_3 are smooth functions on V. Since ξ, e, ϕe are linearly independent, we have $k_3\neq 0$ at any point of V. Using (4.5), (4.6) and (4.9) we get following partial differential equations:

$$\frac{\partial k_1}{\partial y}=\frac{A}{2\lambda}k_1-2,\quad \frac{\partial k_2}{\partial y}=\frac{1}{2\lambda}\left[Ak_2-B-1\right],\quad \frac{\partial k_3}{\partial y}=\frac{A}{2\lambda}k_3, \tag{4.10}$$

$$\frac{\partial k_1}{\partial x}=0,\quad \frac{\partial k_2}{\partial x}=2\lambda,\quad \frac{\partial k_3}{\partial x}=0. \tag{4.11}$$

Moreover we know that

$$\frac{\partial \lambda}{\partial x}=0,\quad \frac{\partial \lambda}{\partial y}=0. \tag{4.12}$$

Differentiating the equation $\frac{\partial k_3}{\partial x}=0$ with respect to $\frac{\partial}{\partial y}$ and using $\frac{\partial k_3}{\partial y}=\frac{A}{2\lambda}k_3$, we find

$$0=\frac{\partial^2 k_3}{\partial y\partial x}=\frac{\partial^2 k_3}{\partial x\partial y}=\frac{1}{2\lambda}\frac{\partial A}{\partial x}k_3+\frac{1}{2\lambda}A\frac{\partial k_3}{\partial x}=\frac{1}{2\lambda}\frac{\partial A}{\partial x}k_3.$$

So

$$\frac{\partial A}{\partial x}=0. \tag{4.13}$$

Differentiating $\frac{\partial k_2}{\partial x}=2\lambda$ with respect to $\frac{\partial}{\partial y}$ and using

$$\frac{\partial k_2}{\partial y}=\frac{1}{2\lambda}\left[Ak_2-B-1\right]$$

and the equation (4.13), we prove that

$$\frac{\partial^2 k_2}{\partial y\partial x}=0=\frac{\partial^2 k_2}{\partial x\partial y}=\frac{1}{2\lambda}\left[\frac{\partial A}{\partial x}k_2+A\frac{\partial k_2}{\partial x}-\frac{\partial B}{\partial x}\right].$$

So

$$\frac{\partial B}{\partial x}=2\lambda A. \tag{4.14}$$

From (4.12) we have following solution

$$\lambda=\hat{g}(z)+d=\breve{g}(z), \tag{4.15}$$

where d is constant. Using $e(\lambda) = k_1\frac{\partial\lambda}{\partial x} + k_2\frac{\partial\lambda}{\partial y} + k_3\frac{\partial\lambda}{\partial z} = 1$ and (4.12), we get

$$\frac{\partial\lambda}{\partial z} = \frac{1}{k_3}, \quad k_3 \neq 0. \tag{4.16}$$

If we differentiate the equation (4.16) with respect to $\frac{\partial}{\partial y}$ and because of the equation $\frac{\partial\lambda}{\partial y} = 0$, we obtain

$$0 = \frac{\partial^2\lambda}{\partial z\partial y} = \frac{\partial^2\lambda}{\partial y\partial z} = -\frac{1}{k_3^2}\frac{\partial k_3}{\partial y}. \tag{4.17}$$

Since $k_3 \neq 0$, the equation (4.17) is reduced to

$$\frac{\partial k_3}{\partial y} = 0. \tag{4.18}$$

Combining (4.10) and (4.18), we deduced that

$$A = 0. \tag{4.19}$$

Using (4.14) and (4.19), we have

$$\frac{\partial B}{\partial x} = 0. \tag{4.20}$$

It follows from (4.20) that

$$B = F(y, z). \tag{4.21}$$

By virtue of (4.19), (4.10) and (4.11), we easily see that

$$k_1 = -2y + r(z), \tag{4.22}$$

where $r(z)$ is integration function. Combining (4.11) and (4.18), we get

$$k_3 = t(z) + \delta, \tag{4.23}$$

where δ is constant. If we use (4.11), (4.15), (4.19) and (4.21) in (4.10)

$$\frac{\partial k_2}{\partial x} = 2\breve{g}(z), \quad \frac{\partial k_2}{\partial y} = \frac{-(B+1)}{2\lambda} = \frac{-(F(y,z)+1)}{2\breve{g}(z)}. \tag{4.24}$$

It follows from this last partial differential equation that

$$k_2 = 2x\breve{g}(z) - \frac{(H(y,z)+y)}{2\breve{g}(z)} + \beta(z), \tag{4.25}$$

where

$$\frac{\partial H(y,z)}{\partial y} = F(y,z). \tag{4.26}$$

Because of (4.16), there is a relation between $\lambda = \breve{g}(z)$ and $k_3(z)$ such that $\breve{g}(z) = \int \frac{1}{k_3(z)} dz$. We will calculate the tensor fields η, ϕ, g with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. For the components g_{ij} of the Riemannian metric g, using (4.9) we have

$$g_{11} = 1,\ g_{22} = 1,\ g_{12} = g_{21} = 0,$$
$$g_{13} = g_{31} = \frac{-k_1}{k_3},$$
$$g_{23} = g_{32} = \frac{-k_2}{k_3},\ g_{33} = \frac{1 + k_1^2 + k_2^2}{k_3^2}.$$

The components of the tensor field ϕ are immediate consequences of

$$\phi(\xi) = \phi(\frac{\partial}{\partial x}) = 0, \quad \phi(\frac{\partial}{\partial y}) = -k_1\frac{\partial}{\partial x} - k_2\frac{\partial}{\partial y} - k_3\frac{\partial}{\partial z},$$
$$\phi(\frac{\partial}{\partial z}) = \frac{k_1 k_2}{k_3}\frac{\partial}{\partial x} + \frac{1 + k_2^2}{k_3}\frac{\partial}{\partial y} + k_2\frac{\partial}{\partial z}.$$

The expression of the contact form η, immediately follows from

$$\eta = dx - \frac{k_1}{k_3}dz.$$

Now we calculate the components of tensor field h with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$h(\xi) = h(\frac{\partial}{\partial x}) = 0, \quad h(\frac{\partial}{\partial y}) = -\lambda\frac{\partial}{\partial y},$$
$$h(\frac{\partial}{\partial z}) = \lambda\frac{k_1}{k_3}\frac{\partial}{\partial x} + 2\lambda\frac{k_2}{k_3}\frac{\partial}{\partial y} + \lambda\frac{\partial}{\partial z}.$$

Case 2: Now we suppose that $p \in U''$. As in Case 1, we can assume that $(\phi e)(\lambda) = 1$. Using the equations (3.7), (3.8), (3.9) and (3.5), (3.6) are reduced

$$[e, \phi e] = -be + c\phi e + 2\xi, \tag{4.27}$$
$$[e, \xi] = 0, \tag{4.28}$$
$$[\phi e, \xi] = -2\lambda e, \tag{4.29}$$

$$b = \frac{(A+1)}{2\lambda}, \quad c = \frac{B}{2\lambda},\ a = -1 - \lambda \tag{4.30}$$

respectively.

Because of (4.28) we find that there exist a chart $\{V',(x, y, z)\}$ at $p \in U''$,

$$\xi = \frac{\partial}{\partial x}, \quad \varphi e = k_1'\frac{\partial}{\partial x} + k_2'\frac{\partial}{\partial y} + k_3'\frac{\partial}{\partial z}, \quad e = \frac{\partial}{\partial y} \tag{4.31}$$

where k_1', k_2' and k_3' ($k_3' \neq 0$), are smooth functions on V'.

Using (4.27), (4.29) and (4.31) we get following partial differential equations:

$$\frac{\partial k_1'}{\partial y} = \frac{B}{2\lambda}k_1' + 2, \quad \frac{\partial k_2'}{\partial y} = \frac{1}{2\lambda}\left[Bk_2' - A - 1\right], \quad \frac{\partial k_3'}{\partial y} = \frac{B}{2\lambda}k_3', \tag{4.32}$$

$$\frac{\partial k_1'}{\partial x} = 0, \quad \frac{\partial k_2'}{\partial x} = 2\lambda, \quad \frac{\partial k_3'}{\partial x} = 0. \tag{4.33}$$

Moreover we know that

$$\frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial \lambda}{\partial y} = 0. \tag{4.34}$$

As in Case 1, if we solve partial differential equations (4.32), (4.33) and (4.34), then we find

$$B = 0, \quad A = F(y,z), \tag{4.35}$$

$$\lambda = \bar{g}(z) + d' = g(z), \quad k_1' = 2y + r'(z), \quad k_3' = t'(z) + \delta', \tag{4.36}$$

$$k_2' = 2xg(z) - \frac{(H(y,z) + y)}{2g(z)} + \beta'(z), \tag{4.37}$$

$$\frac{\partial H(y,z)}{\partial y} = F(y,z), \tag{4.38}$$

where $r'(z)$ is integration function, d' and δ' are constants. By the help of (4.36), the equation $(\phi e)(\lambda) = 1$ implies that

$$\lambda(z) = g(z) = \int \frac{1}{k_3'(z)} dz. \tag{4.39}$$

As in Case 1, we can directly calculate the tensor fields g, ϕ, η and h with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$g = \begin{pmatrix} 1 & 0 & -\frac{k_1'}{k_3'} \\ 0 & 1 & -\frac{k_2'}{k_3'} \\ -\frac{k_1'}{k_3'} & -\frac{k_2'}{k_3'} & \frac{1+k_1'^2+k_2'^2}{k_3'^2} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & k_1' & -\frac{k_1' k_2'}{k_3'} \\ 0 & k_2' & -\frac{1+k_2'^2}{k_3'} \\ 0 & k_3' & -k_2' \end{pmatrix},$$

$$\eta = dx - \frac{k_1'}{k_3'} dz \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & -\lambda \frac{k_1'}{k_3'} \\ 0 & \lambda & -2\lambda \frac{k_2'}{k_3'} \\ 0 & 0 & -\lambda \end{pmatrix}.$$

□

Example. We consider the 3-dimensional manifold

$$M = \{(x,y,z) \in R^3, z \neq 0\}$$

and the vector fields

$$\xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y}, \quad e = -2y\frac{\partial}{\partial x} + (2xz - 1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form $\eta = dx + 2y dz$ defines a contact structure on M with characteristic vector field $\xi = \frac{\partial}{\partial x}$. Let g, ϕ be the Riemannian metric and the $(1,1)$-tensor field given by

$$g = \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ a_1 & a_2 & 1+a_1^2+a_2^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & a_1 & a_1 a_2 \\ 0 & a_2 & a_2^2+1 \\ 0 & -1 & -a_2 \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & 0 & -2yz \\ 0 & -z & 2z(2xz-1) \\ 0 & 0 & z \end{pmatrix}, \quad \lambda = z,$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, where $a_1 = 2y$ and $a_2 = 1-2xz$. By a straightforward calculation, we obtain

$$\nabla_\xi \tau = 2(z-1)\tau\phi.$$

Now, we will give an example which satisfies the conditions of Theorem 4.1.

Example. We consider the 3-dimensional manifold

$$M = \{(x,y,z) \in R^3, z > 0\}$$

and the vector fields

$$\xi = \frac{\partial}{\partial x}, \quad e = \frac{\partial}{\partial y}, \quad \phi e = 2y\frac{\partial}{\partial x} + (2xz - \frac{2z+y}{2z})\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

The 1-form $\eta = dx - \frac{2y}{z}dz$ defines a contact structure on M with characteristic vector field $\xi = \frac{\partial}{\partial x}$. Let g, ϕ be the Riemannian metric and the $(1,1)$-tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \\ -\frac{a_1}{a_3} & -\frac{a_2}{a_3} & \frac{1+a_1^2+a_2^2}{a_3^2} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & a_1 & -\frac{a_1 a_2}{a_3} \\ 0 & a_2 & -\frac{1+a_2^2}{a_3} \\ 0 & a_3 & -a_2 \end{pmatrix},$$

$$\eta = dx - \frac{a_1}{a_3}dz \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & -\lambda\frac{a_1}{a_3} \\ 0 & \lambda & -2\lambda\frac{a_2}{a_3} \\ 0 & 0 & -\lambda \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, where $a_1 = 2y$, $a_2 = 2xz - \frac{2z+y}{2z}$, $a_3 = z$ and $\lambda = \ln(z)$. By direct computations, we get

$$\begin{aligned} &R(X,Y)\xi \\ &= (1-(\ln(z))^2)(\eta(Y)X - \eta(X)Y) + 2(-1-\ln(z))(\eta(Y)hX - \eta(X)hY) \end{aligned}$$

and

$$\nabla_\xi \tau = 2(-\ln(z) - 1)\tau\phi.$$

5. Some Applications

In this section, we will give several properties and some applications about contact metric 3-manifold with $\|grad\ \lambda\| = 1$ and $\nabla_\xi \tau = 2a\tau\phi$. This class of manifold is denoted by Ω.

Remark 5.1. Since the Case 2 ($A = F(y,z), B = 0, a = \lambda - 1, b = 0$) is similar to the Case 1 ($A = 0, B = F(y,z), a = -\lambda - 1, c = 0$), we only discuss the Case 1. By virtue of $\|grad\ \lambda\| \neq 0$ we can conclude that Ω is neither Sasakian nor flat.

Remark 5.2. Calvaruso and Perrone [7] proved that a semi-symmetric contact metric 3-manifold satisfying $A = 0$ or $B = 0$, either is flat or has constant curvature 1. Hence Ω is not semi-symmetric space.

5.1. Harmonic vector fields

If (M, g) is a Riemannian manifold and (T^1M, g_s) is its unit tangent sphere bundle equipped with the Sasaki metric g_s, a unit vector field V on M determines a map between (M, g) and (T^1M, g_s). When M is compact and orientable, the energy of V is the energy $E(V) = \frac{1}{2}\int_M \|dV\|^2 dv = \frac{m}{2} vol$ $(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 dv$ of the corresponding map. V is said to be a harmonic vector field if it is a critical point for the energy functional E defined on the space $\chi^1(M)$ unit vector fields on (M, g).

By an H-contact manifold [20] we mean a contact metric manifold such that the characteristic vector field ξ is harmonic, that is ξ is an eigenvector of the rough Laplacian Δ. It was shown in [20] that M is an H-contact manifold if and only if ξ is an eigenvector of the Ricci operator.

In [16] Koufogiorgos and Stamatiou proved that every contact metric manifold M satisfying $R(X,Y)\xi = 0$, for any $X, Y \in D = ker\eta$ is an H-contact manifold. Koufogiorgos et al. proved following Theorems:

Theorem 5.3. [14] *Let M be a 3-dimensional contact metric manifold. If M is an H-contact manifold, then M is a (κ, μ, ν)-contact metric manifold on an everywhere open and dense subset of M.*

Theorem 5.4. [15] *Let M be a 3-dimensional (κ, μ, υ)-contact metric manifold for which $\nabla_\xi \tau = 2a\tau\phi$ where a is smooth funtion on M. Then M is either a Sasakian manifold or generalized (κ, μ)-contact metric manifold.*

If we put an extra assumption $F(y,z) = 0$ relative to this chart then ξ is an eigenvector of the Ricci operator. So Ω becomes an H-contact manifold. Hence we obtain

Corollary 5.5. *If $F = 0$, then Ω is a generalized (κ, μ)-contact metric manifold. In particular, Ω is H-contact.*

5.2. Strongly locally ϕ-symmetry

M is called a strongly locally ϕ-symmetric contact metric manifold if the characteristic reflections are local isometries ([8], [16]).

Calvaruso et al. [8] proved that a contact metric 3-manifold M strongly locally ϕ-symmetric spaces if and only if $\tau = 0$ and scalar curvature ρ is constant or M is a (κ, μ)-contact metric manifold. Using this result and (2.10) we obtain

Corollary 5.6. *Ω is not strongly locally ϕ-symmetric space.*

5.3. Conformal flatness

A 3-dimensional Riemannian manifold M is called conformally flat if and only if the Ricci operator Q satisfies

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}\{(X(\rho)Y - (Y(\rho)X\} \tag{5.1}$$

for all vector fields X and Y.

Lemma 5.7. *Ω is a conformally flat if and only if*

$$\begin{aligned}
&\text{i) } \xi(\rho) = 0, \ \text{ ii) } \phi e(\rho) = 0, \ \text{ iii) } \xi(B) = 0, \ \text{ iv) } \phi e(B) = 0,\\
&\text{v) } \frac{e(\rho)}{4} = B(1+3\lambda) - 4\lambda, \ \text{ vi) } e(B) + 9\lambda^3 - 5\lambda^2 + \lambda(\frac{\rho}{2} - 1) - 3 + \frac{\rho}{2} = 0,\\
&\text{vii) } B(\frac{B+1}{2\lambda}) + 5\lambda^3 - 7\lambda^2 + \lambda(\frac{\rho}{2} - 1) + 3 - \frac{\rho}{2} = 0,\\
&\text{viii) } e(B) - B(\frac{B+1}{2\lambda}) + 4\lambda^3 + 2\lambda^2 - 6 + \rho = 0.
\end{aligned} \tag{5.2}$$

On the other hand, Gouli-Andreou et.al [10] investigated conformally flat 3-τ-a manifolds and proved the following Theorem.

Theorem 5.8. [10] *Let M be a 3-dimensional conformally flat 3-τ-a manifold with a and scalar curvature ρ smooth functions, constant along the flow of ξ. Then M is either flat or Sasakian with constant curvature 1 or a semi-K contact manifold.*

By virtue of Definition 3.2 ,Theorem 4.1, Lemma 5.7 and Theorem 5.8 we get

Corollary 5.9. *Let Ω be a conformally flat space. Then Ω is a semi-K contact manifold.*

Remark 5.10. We suppose that $A = B = 0$. Under this condition Calvaruso et.al. [8] proved that a conformally flat contact metric 3-manifold has constant sectional curvature 0 or 1. Then M is either flat or Sasakian with constant curvature 1.

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