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New families of special numbers and polynomials arising from applications of p-adic q-integrals

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Abstract

In this manuscript, generating functions are constructed for the new special families of polynomials and numbers using the p-adic q-integral technique. Partial derivative equations, functional equations and other properties of these generating functions are given. With the help of these equations, many interesting and useful identities, relations, and formulas are derived. We also give p-adic q-integral representations of these numbers and polynomials. The results we have obtained for these special numbers and polynomials are closely related to well-known families of polynomials and numbers including the Bernoulli numbers, the Apostol-type Bernoulli numbers and polynomials and the Frobenius-Euler numbers, the Stirling numbers, and the Daehee numbers. We give some remarks and observations on the results of this paper.

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1 Introduction

It is well known that special numbers and polynomials related to the Bernoulli numbers and polynomials, Euler type numbers and polynomials, the Stirling numbers have many applications in mathematics and in other areas. Recently, there were many special numbers and polynomials having applications in not only mathematics, but also in physics, in computer science, in engineering and in the analysis of ambulatory blood pressure measurements and also biostatistics problems. Polynomials are used to model real phenomena in these areas (*cf.* [1–29]; see also the references cited therein).

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ are defined by

$$F_A(t,x;\lambda) = \frac{t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{t^n}{n!}. \tag{1.1}$$

One has the following the Apostol-Bernoulli numbers:

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$$

(*cf.* [1–29]; see also the references cited therein).

Kim *et al.* [15] gave the following new type Apostol polynomials $\mathfrak{B}_n(x;\lambda)$:

$$F_B(t,x;\lambda) = \frac{\log\lambda + t}{\lambda e^t - 1}e^{tx} = \sum_{n=0}^{\infty}\mathfrak{B}_n(x;\lambda)\frac{t^n}{n!} \tag{1.2}$$

(*cf.* [15, 28]).

The Frobenius-Euler numbers $H_n(u)$ are given by

$$F_f(t,u) = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty}H_n(u)\frac{t^n}{n!},$$

where $u \neq 1$ (*cf.* [10], [15], Theorem 1, p.439, [21, 28]).

The λ-Stirling numbers $S_2(n,k;\lambda)$ are defined by

$$F_S(t,k;\lambda) = \frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty}S_2(n,k;\lambda)\frac{t^n}{n!}, \tag{1.3}$$

where $k \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ (*cf.* [17, 22, 27]).

Substituting $\lambda = 1$ into (1.3), one has the following classical Stirling numbers of the second kind:

$$S_2(n,v) = S_2(n,v;1)$$

(*cf.* [1–29]; see also the references cited therein).

The Daehee numbers of the first kind are defined by

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty}D_n\frac{t^n}{n!}$$

(*cf.* [7]). The Daehee numbers are related to factorial numbers, this explicit relation is given by

$$D_n = (-1)^n\frac{n!}{n+1}$$

(*cf.* [19], p.45, [7, 24, 25]).

We briefly summarize the main results of this manuscript as follows:

In Section 2, by using the p-adic q-integral and its integral equation, we construct generating functions for special numbers and polynomials. We give some properties of these numbers and polynomials. By using functional equations of these generating functions, we also give some formulas and relations for these numbers and polynomials.

In Section 3, we give partial differential equations for the generating functions. By using these equations, we derive some derivative formulas for these numbers and polynomials.

In Section 4, we give a conclusion of this paper.

2 Construction of generating functions by the *p*-adic *q*-integral

In this section, by using the p-adic integral on $\mathbb{Z}_p$, we construct generating functions for a new family of numbers and polynomials which are related to the Apostol-type numbers and polynomials. We give relations between these numbers, the λ-Bernoulli numbers, the Stirling numbers and the Daehee numbers. We also give p-adic integral representation of these numbers. Firstly we give some standard notations for the Volkenborn integral. Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{K}$ be a field with a complete valuation, $\mathbb{K} \supseteq \mathbb{Q}_p$ and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be a set of continuously differentiable functions. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with

$$|p|_p = p^{-v_p(p)} = 1/p.$$

It is well known that q is considered as an indeterminate. That is, if $q \in \mathbb{C}$, then $|q| < 1$ and also if $q \in \mathbb{C}_p$, then

$$|q-1|_p < p^{-\frac{1}{p-1}}.$$

On the other hand, if $|x|_p \leq 1$, then

$$q^x = \exp(x \log q).$$

Definition 1 The Volkenborn integral of the function $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$ is

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{2.1}$$

where μ_1 denotes the Haar distribution defined by

$$\mu_1\bigl(x + p^N \mathbb{Z}_p\bigr) = \frac{1}{p^N}$$

(*cf.* [11, 20]).

Throughout this paper, p is an odd prime.

Kim [11] gave the q-Haar distribution $\mu_q(x) = \mu_q(x + p^N \mathbb{Z}_p)$ and the p-adic q-integral $I_q(f)$ as follows, respectively:

$$\mu_q\bigl(x + p^N \mathbb{Z}_p\bigr) = \frac{q^x}{[p^N]_q},$$

where $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{2.2}$$

where $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$ and

$$[x] = [x:q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1, \\ x, & q = 1. \end{cases}$$

The integral equation of (2.2) was derived by Kim [14] as follows:

$$q\int_{\mathbb{Z}_p} f(x+1)\,d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) + q\frac{q-1}{\log q}f'(0) + q(q-1)f(0), \tag{2.3}$$

where

$$f'(0) = \frac{d}{dx}f(x)\bigg|_{x=0}.$$

In [20], p.70, exponential function is defined by

$$e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!}.$$

The above series converges in region E, which is a subset of field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Let k be residue class field of $\mathbb{K}$. If $\mathrm{char}(k) = p$, then

$$E = \left\{x \in \mathbb{K} : |x| < p^{\frac{1}{1-p}}\right\}$$

and if $\mathrm{char}(k) = 0$, then

$$E = \left\{x \in \mathbb{K} : |x| < 1\right\}.$$

Substituting the following function:

$$f(x) = a^{xt}$$

into (2.3), we obtain

$$\int_{\mathbb{Z}_p} a^{xt}\,d\mu_q(x) = \left(\frac{t\log a}{\log q} + 1\right)\frac{q(q-1)}{qa^t - 1}, \tag{2.4}$$

where $a \neq 1$ and

$$a \in \mathbb{C}_p^+ = \left\{x \in \mathbb{C}_p : |1-x|_p < 1\right\}.$$

If $t = 1$ and $q \to 1$ into (2.4), we derive Exercise 55A-1 in [20], p.170, as follows:

$$\int_{\mathbb{Z}_p} a^x\,d\mu_1(x) = \frac{\log_p(a)}{a-1},$$

where $a \in \mathbb{C}_p^+$ and $a \neq 1$.

If $a = e$ and $q \to 1$ into (2.4), we also derive Exercise 55A-2 in [20], p.170, as follows:

$$\int_{\mathbb{Z}_p} e^{tx}\,d\mu_1(x) = \frac{t}{e^t - 1},$$

where $t \in E$ with $t \neq 0$. The right hand side of the above equation gives us the generating function for the Bernoulli numbers B_n.

By using (2.4), we construct the following generating function, which is related to the Apostol-type Bernoulli numbers:

$$F(t;a,q)=\left(\frac{t\log a}{\log q}+1\right)\frac{q(q-1)}{qa^t-1}=\sum_{n=0}^{\infty}\mathfrak{s}_n(a;q)\frac{t^n}{n!}. \tag{2.5}$$

By using (2.5), we give a recurrence relation for the Apostol-type Bernoulli numbers as follows:

$$\begin{aligned}q(q-1)\frac{\log a}{\log q}t+q(q-1)&=q\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(\log a)^{n-j}\mathfrak{s}_j(a;q)\right)\frac{t^n}{n!}\\&\quad-\sum_{n=0}^{\infty}\mathfrak{s}_n(a;q)\frac{t^n}{n!}.\end{aligned}$$

If we equate the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, then we get the following theorem.

Theorem 1 *Let*

$$\mathfrak{s}_0(a;q)=q$$

and

$$\mathfrak{s}_1(a;q)=\frac{(q-1-q\log q)q\log a}{(q-1)\log q}.$$

If $n\geq 2$, *we have*

$$\mathfrak{s}_n(a;q)=q\sum_{j=0}^{n}\binom{n}{j}(\log a)^{n-j}\mathfrak{s}_j(a;q).$$

We are ready to give generating function for Apostol-type Bernoulli polynomials as follows:

$$G(t,x;a,b,q)=a^{tx}F(t;a,q)=\sum_{n=0}^{\infty}\mathfrak{s}_n(x;a;q)\frac{t^n}{n!}. \tag{2.6}$$

By using (2.6), we obtain

$$\sum_{n=0}^{\infty}\mathfrak{s}_n(x+y;a;q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathfrak{s}_n(x;a;q)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{(ty\log a)^n}{n!}.$$

By using the Cauchy product in the above equation, we obtain

$$\sum_{n=0}^{\infty}\mathfrak{s}_n(x+y;a;q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(y\log a)^{n-j}\mathfrak{s}_j(x;a;q)\right)\frac{t^n}{n!}.$$

If we equate the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, then we get the following theorem.

Theorem 2 *The following formula holds*:

$$\mathfrak{s}_n(x+y;a;q) = \sum_{j=0}^{n} \binom{n}{j} (y\log a)^{n-j} \mathfrak{s}_j(x;a;q). \tag{2.7}$$

Substituting $y = -1$ into (2.7), we get

$$\mathfrak{s}_n(x-1;a;q) = \sum_{j=0}^{n} \binom{n}{j} (-\log a)^{n-j} \mathfrak{s}_j(x;a;q).$$

The main motivation of the generating function $F(t;a,q)$ is given as follows:

$$q\left(\frac{t\log a}{\log q} + 1\right) \sum_{n=0}^{\infty} (\log a)^n H_n\left(\frac{1}{q}\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{s}_n(a;q) \frac{t^n}{n!}.$$

From the above functional equation, we get

$$\begin{aligned} &q\frac{\log a}{\log q} \sum_{n=0}^{\infty} n(\log a)^{n-1} H_{n-1}\left(\frac{1}{q}\right) \frac{t^n}{n!} + q\sum_{n=0}^{\infty} (\log a)^n H_n\left(\frac{1}{q}\right) \frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty} \mathfrak{s}_n(a;q) \frac{t^n}{n!}. \end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get a relation between the Frobenius-Euler numbers and the numbers $\mathfrak{s}_n(a;q)$ by the following theorem.

Theorem 3 *The following formula holds*:

$$\mathfrak{s}_n(a;q) = qn\frac{\log a}{\log q} (\log a)^{n-1} H_{n-1}\left(\frac{1}{q}\right) + q(\log a)^n H_n\left(\frac{1}{q}\right).$$

By combining (2.4) and (2.5), we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{Z}_p} (x\log a)^n \, d\mu_q(x) = \sum_{n=0}^{\infty} \mathfrak{s}_n(a;q) \frac{t^n}{n!}.$$

By using the above equation, we get a p-adic integral representation for the numbers $\mathfrak{s}_n(a;q)$ as follows.

Theorem 4 *The following identity holds*:

$$\mathfrak{s}_n(a;q) = \int_{\mathbb{Z}_p} (x\log a)^n \, d\mu_q(x). \tag{2.8}$$

Remark 1 Setting $q \to 1$ and $a = e$ in (2.8), we arrive at the Witt formula for the Bernoulli numbers:

$$\int_{\mathbb{Z}_p} x^n \, d\mu_1(x) = B_n. \tag{2.9}$$

We also easily see that

$$\int_{\mathbb{Z}_p} (z+x)^n \, d\mu_1(x) = B_n(z) \tag{2.10}$$

(*cf.* [11, 13, 20]; see also the references cited therein).

Combining (2.4) and (2.5), we get

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (\log a)^n S_2(n,m) \int_{\mathbb{Z}_p} (x)_m \, d\mu_q(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{s}_n(a;q) \frac{t^n}{n!}.$$

If we equate the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, then we get the following theorem.

Theorem 5 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{s}_n(a;q) = \sum_{m=0}^{n} (\log a)^n S_2(n,m) \int_{\mathbb{Z}_p} (x)_m \, d\mu_q(x). \tag{2.11}$$

The q-Daehee numbers are defined by means of the following p-adic q-integral representation:

$$D_m(q) = \int_{\mathbb{Z}_p} (x)_m \, d\mu_q(x) \tag{2.12}$$

(*cf.* [7, 9]). Substituting (2.12) into (2.11), we get a relation between the numbers $\mathfrak{s}_n(a;q)$ and the numbers $D_m(q)$ by the following theorem.

Theorem 6 *The following formula holds*:

$$\mathfrak{s}_n(a;q) = \sum_{m=0}^{n} (\log a)^n S_2(n,m) D_m(q). \tag{2.13}$$

Remark 2 Substituting $q \to 1$ into (2.11), we get

$$\mathfrak{s}_n(a;1) = \sum_{m=0}^{n} (\log a)^n S_2(n,m) D_m.$$

Remark 3 Substituting $a = e$ into (2.13), we get

$$B_n = \sum_{m=0}^{n} S_2(n,m) D_m$$

(*cf.* [7, 24, 26]).

3 Partial differential equations for generating functions

In this section, we give partial differential equations for the generating functions.

We compute the derivative of equation (2.6) with respect to x to derive the following higher order partial differential equation:

$$\frac{\partial^k}{\partial x^k}\left\{G(t,x;a,b,q)\right\} = (t\log a)^k G(t,x;a,b,q),$$

where k is a nonnegative integer. By combining the above equation with (2.6) we get

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial x^k}\left\{\mathfrak{s}_n(x;a;q)\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\log a)^k \mathfrak{s}_n(x;a;q)\frac{t^{n+k}}{n!}.$$

After some calculation, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem.

Theorem 7

$$\frac{\partial^k}{\partial x^k}\left\{\mathfrak{s}_n(x;a;q)\right\} = k!\binom{n}{k}(\log a)^k \mathfrak{s}_{n-k}(x;a;q).$$

We also compute the derivative of equation (2.5) with respect to t and q to derive the following partial differential equations, respectively:

$$t\frac{\partial}{\partial t}\left\{F(t;a,q)\right\} = \frac{q(q-1)}{\log q}F_A(t\log a,0;q) - qF_A(t\log a,1;q)F(t;a,q) \tag{3.1}$$

and

$$\begin{aligned} t\frac{\partial}{\partial q}\left\{F(t;a,q)\right\} &= t\frac{2q+(2q-1)\log q}{(\log q)^2}F_A(t\log a,0;q) + \frac{1}{\log a}F_A(t\log a,0;q) \\ &\quad - \frac{1}{\log a}F_A(t\log a,1;q)F(t;a,q), \end{aligned} \tag{3.2}$$

where $a \neq 1$.

Theorem 8 *Let n be a nonnegative integer. Then we have*

$$n\mathfrak{s}_n(a;q) = \frac{q(q-1)}{\log q}(\log a)^n \mathcal{B}_n(q) - q\sum_{j=0}^{n}\binom{n}{j}(\log a)^{n-j}\mathcal{B}_{n-j}(1;q)\mathfrak{s}_j(a;q).$$

Proof Combining (1.1) and (2.5) with (3.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty}\mathfrak{s}_n(a;q)\frac{t^n}{n!} &= \frac{q(q-1)}{\log q}\sum_{n=0}^{\infty}(\log a)^n \mathcal{B}_n(q)\frac{t^n}{n!} \\ &\quad - q\sum_{n=0}^{\infty}(\log a)^n \mathcal{B}_n(1;q)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{s}_n(a;q)\frac{t^n}{n!}. \end{aligned}$$

Using the Cauchy product into the above equation, after some elementary calculation, we obtain

$$\sum_{n=0}^{\infty}\mathfrak{s}_n(a;q)\frac{t^n}{n!}=\frac{q(q-1)}{\log q}\sum_{n=0}^{\infty}(\log a)^n\mathcal{B}_n(q)\frac{t^n}{n!}$$
$$-q\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(\log a)^{n-j}\mathcal{B}_{n-j}(1;q)\mathfrak{s}_j(a;q)\frac{t^n}{n!}.$$

If we equate the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, then we get the desired result. □

Theorem 9 *Let n be a positive integer and $a \neq 1$. Then we have*

$$n\frac{\partial}{\partial q}\left\{\mathfrak{s}_{n-1}(a;q)\right\}=\frac{2q+(2q-1)\log q}{(\log q)^2}(\log a)^{n-1}\mathcal{B}_{n-1}(q)-\frac{(2q-1)(\log a)^n\mathcal{B}_n(q)}{\log a}$$
$$-\frac{1}{\log a}\sum_{j=0}^{n}\binom{n}{j}(\log a)^{n-j}\mathcal{B}_{n-j}(1;q)\mathfrak{s}_j(a;q).$$

Proof Combining (1.1) and (2.5) with (3.2), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}\left\{\mathfrak{s}_n(a;q)\right\}\frac{t^{n+1}}{n!}=\frac{2q+(2q-1)\log q}{(\log q)^2}\sum_{n=0}^{\infty}(\log a)^n\mathcal{B}_n(q)\frac{t^{n+1}}{n!}$$
$$-\frac{1}{\log a}\sum_{n=0}^{\infty}(\log a)^n\mathcal{B}_n(1;q)\frac{t^n}{n!}$$
$$-\frac{1}{\log a}\sum_{n=0}^{\infty}(\log a)^n\mathcal{B}_n(1;q)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{s}_n(a;q)\frac{t^n}{n!}.$$

Using the Cauchy product into the above equation, after some elementary calculation, we obtain

$$\sum_{n=0}^{\infty}n\frac{\partial}{\partial q}\left\{\mathfrak{s}_{n-1}(a;q)\right\}\frac{t^n}{n!}=\frac{2q+(2q-1)\log q}{(\log q)^2}\sum_{n=0}^{\infty}n(\log a)^{n-1}\mathcal{B}_n(q)\frac{t^n}{n!}$$
$$-\frac{1}{\log a}\sum_{n=0}^{\infty}(\log a)^n\mathcal{B}_n(1;q)\frac{t^n}{n!}$$
$$-\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(\log a)^{n-j}\mathcal{B}_{n-j}(1;q)\mathfrak{s}_j(a;q)\frac{t^n}{n!}.$$

If we equate the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, then we get the desired result. □

4 Conclusion

By applying the p-adic q-integral equation to continuously differentiable functions on ring of p-adic integers, in this manuscript we define generating functions for new family of special numbers and polynomials. Using these generating functions and their functional equations, we derive many new identities and relations for these new families of special numbers and polynomials. Special values of these numbers and polynomials are associated with the Apostol-type Bernoulli numbers and polynomials and the Frobenius-Euler numbers. Moreover, we give partial differential equations for these generating functions. By using these equations, we derive some identities and derivative formulas for these new numbers and polynomials. In addition, the results of this manuscript will shed light on many branches of mathematics, physics, engineering, and other sciences.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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