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ON GENERALIZED ROTATIONAL SURFACES IN EUCLIDEAN SPACES

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Abstract. In the present study we consider the generalized rotational surfaces in Euclidean spaces. Firstly, we consider generalized tractrices in Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. Further, we introduce some kind of generalized rotational surfaces in Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$, respectively. We have also obtained some basic properties of generalized rotational surfaces in $\mathbb{E}^4$ and some results of their curvatures. Finally, we give some examples of generalized Beltrami surfaces in $\mathbb{E}^3$ and $\mathbb{E}^4$, respectively.

1. Introduction

The Gaussian curvature and mean curvature of the surfaces in Euclidean spaces play an important role in differential geometry. Especially, surfaces with constant Gaussian curvature [19], and constant mean curvature conform nice classes of surfaces which are important for surface modelling [7]. Surfaces with constant negative curvature are known as pseudo-spherical surfaces (see [16]).

Rotational surfaces in Euclidean spaces are also important subject of differential geometry. The rotational surfaces in $\mathbb{E}^3$ are called surfaces of revolution. Recently V. Velickovic classified all rotational surfaces in $\mathbb{E}^3$ with constant Gaussian curvature [18]. Rotational surfaces in $\mathbb{E}^4$ was first introduced by C. Moore in 1919. In the recent years some mathematicians have taken an interest in the rotational surfaces in $\mathbb{E}^4$, for example G. Ganchev and V. Milousheva [14], U. Dursun and N. C. Turgay [13], the first author et al. [3]. In [14], the authors applied the invariance theory of surfaces in the four dimensional Euclidean space to the class of general rotational surfaces whose meridians lie in two dimensional planes in order to find all minimal surfaces. See also [2] for rotational surfaces have pointwise 1-type Gauss map in $\mathbb{E}^4$. The first author et al. in [3] gave the necessary and sufficient conditions for generalized rotation surfaces to become pseudo-umbilical. They also gave some special classes

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of generalized rotational surfaces as examples. See also [6], [12] and [20] for the rotational surfaces with constant Gaussian curvature in Euclidean 4-space $\mathbb{E}^4$. For higher dimensional case N. H. Kuiper defined rotational embedded submanifolds in Euclidean spaces [17].

In [16] V. A. Gorkavyi and E. N. Nevmerzhitskaya introduced a special class of curves in $\mathbb{E}^n$ called generalized tractrices. Then, by applying special motions in $\mathbb{E}^n$ to generalized tractrices, they construct a special class of pseudo-spherical surfaces in $\mathbb{E}^n$ called generalized Beltrami surfaces.

This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in $\mathbb{E}^n$. Section 3 explains some geometric properties of generalized tractrices $\mathbb{E}^{n+1}$. Section 4 tells about the generalized rotational surfaces in $\mathbb{E}^{n+m}$. Further, this section provides some basic properties of generalized rotational surfaces in $\mathbb{E}^4$ and some results of their curvatures. We also shown that every generalized Beltrami surfaces in $\mathbb{E}^4$ have constant Gaussian curvature $K = -1/c^2$. Finally, we present some examples of generalized Beltrami surfaces in $\mathbb{E}^4$.

2. Basic concepts

Let M be a smooth surface in $\mathbb{E}^n$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$g_{11} = \langle X_u, X_u \rangle, g_{12} = \langle X_u, X_v \rangle, g_{22} = \langle X_v, X_v \rangle, \tag{2.1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = g_{11}g_{22} - g_{12}^2 \neq 0$, i.e., the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^n$.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Given any local vector fields X_1, X_2 tangent to M, consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i,j \leq 2, \tag{2.2}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{E}^n$, respectively. This map is well-defined, symmetric and bilinear [9].

For any arbitrary orthonormal frame field $\{N_1, N_2, \ldots, N_{n-2}\}$ of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_k} X_j = -(\widetilde{\nabla}_{X_j} N_k)^T, \quad X_j \in \chi(M). \tag{2.3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = L_{ij}^k, \quad 1 \leq i,j \leq 2; \quad 1 \leq k \leq n-2, \tag{2.4}$$

where L_{ij}^k are the coefficients of the second fundamental form. The equation (2.2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{n-2} L_{ij}^k N_k, \quad 1 \leq i, j \leq 2 \tag{2.5}$$

hold. Then the *Gauss curvature* K of a regular patch $X(u,v)$ is given by

$$K = \frac{1}{W^2} \sum_{k=1}^{n-2} (L_{11}^k L_{22}^k - (L_{12}^k)^2). \tag{2.6}$$

Further, the *mean curvature vector* of a regular patch $X(u,v)$ is given by

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12}) N_k. \tag{2.7}$$

We call the functions

$$H_k = \frac{(L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12})}{2W^2} \tag{2.8}$$

the k-th *mean curvature* of the given surface. The norm of the mean curvature vector $H = \left\| \overrightarrow{H} \right\|$ is called the mean curvature of M. Recall that a surface M is said to be *flat* (resp. *minimal*) if its Gauss curvature (resp. mean curvature vector) vanishes identically [8], [10].

The *normal curvature* K_N of M is defined by (see [11])

$$K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} \left\langle R^{\perp}(X_1, X_2) N_\alpha, N_\beta \right\rangle^2 \right\}^{1/2}, \tag{2.9}$$

where

$$R^{\perp}(X_i, X_j) N_\alpha = h(X_i, A_{N_\alpha} X_j) - h(X_j, A_{N_\alpha} X_i), \tag{2.10}$$

and

$$\left\langle R^{\perp}(X_i, X_j) N_\alpha, N_\beta \right\rangle = \left\langle [A_{N_\alpha}, A_{N_\beta}] X_i, X_j \right\rangle \tag{2.11}$$

is called the *equation of Ricci.* We observe that the normal connection D of M is flat if and only if $K_N = 0$, and by a result of Cartan, this equivalent to the diagonalisability of all shape operators A_{N_α} of M which means that M is a *totally umbilical* surface in $\mathbb{E}^n$ [1].

3. Generalized tractrices

Let γ be a regular oriented curve in $\mathbb{E}^{n+1}$ that does not lie in any subspace of $\mathbb{E}^{n+1}$. From each point of the curve γ one can draw a segment of unit length along the tangential line corresponding to the chosen orientation. The ends of these segments describe a new curve β. The curve $\gamma \in \mathbb{E}^{n+1}$ is called a *generalized tractrix* if the curve β lies in a certain subspace $\mathbb{E}^n$of $\mathbb{E}^{n+1}$. The curve β is called the trace of γ [16]. Let

$$\gamma(u) = (f_1(u), \ldots, f_{n+1}(u)) \tag{3.1}$$

be the radius vector of the curve γ given with arc-length parametrization u, i.e., $\|\gamma'(u)\| = 1$. The curve β is defined by the radius vector

$$\beta(u) = (\gamma + c\gamma')(u) = \left((f_1 + cf_1')(u), \ldots, (f_{n+1} + cf_{n+1}')(u)\right), \tag{3.2}$$

where $c > 0$ is a real constant. If γ is a generalized tractrix of $\mathbb{E}^{n+1}$, then by definition the curve β lies in the hyperplane $\mathbb{E}^n$ if and only if $f_{n+1} + cf_{n+1}' = 0$. Consequently, this equation has a non-trivial solution

$$f_{n+1}(u) = \lambda e^{-u/c}, \tag{3.3}$$

where λ is a constant. Thus, the radius vector of the generalized tractrix γ takes the form

$$\gamma(u) = \Big(f_1(u), \ldots, f_n(u), \lambda e^{-u/c}\Big). \tag{3.4}$$

Moreover, the condition for the arc-length parameter u implies that

$$(f_1')^2 + \cdots + (f_n')^2 = 1 - \frac{\lambda^2}{c^2} e^{-2u/c}. \tag{3.5}$$

For convenience, we introduce a vector function

$$\phi(u) = (f_1(u), \ldots, f_n(u); 0).$$

Then the radius vector (3.4) can be represented in the form

$$\gamma(u) = \phi(u) + \lambda e^{-u/c} e_{n+1}, \tag{3.6}$$

where $e_{n+1} = (0, 0, \ldots, 0, 1)$. Consequently, the condition (3.5) gives

$$\|\phi'(u)\|^2 = 1 - \frac{\lambda^2}{c^2} e^{-2u/c}. \tag{3.7}$$

Hence, the radius vector of the trace curve β becomes

$$\beta(u) = \phi(u) + \phi'(u). \tag{3.8}$$

Consider an arbitrary unit vector function

$$a(u) = (a_1(u), \ldots, a_n(u); 0) \tag{3.9}$$

in $\mathbb{E}^{n+1}$ and use this function to construct a new vector function

$$\phi(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} a(u) du \tag{3.10}$$

whose last coordinate is equal to zero. Consequently, it is easy to see that the vector function $\alpha(u)$ satisfies the condition (3.7) and generates a generalized tractrix with radius vector (3.6).

Example 3.1. The ordinary tractrix in $\mathbb{E}^2$ is given with the radius vector

$$\gamma(u) = \left(\int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} du, \lambda e^{-u/c} \right). \tag{3.11}$$

Example 3.2. Consider the unit vector

$$a(u) = (\cos \alpha(u), \sin \alpha(u); 0)$$

in $\mathbb{E}^2$. Then, by the use of (3.10), the corresponding generalized tractrix γ in $\mathbb{E}^3$ is defined by the radius vector

$$\gamma(u) = \left(\int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} \cos \alpha(u) du, \int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} \sin \alpha(u) du; \lambda e^{-u/c} \right). \tag{3.12}$$

Example 3.3. Consider the unit vector

$$a(u) = \big(\cos \alpha(u), \cos \alpha(u) \sin \alpha(u), \sin^2 \alpha(u); 0\big)$$

in $\mathbb{E}^3$. Then using (3.10), the corresponding generalized tractrix γ in $\mathbb{E}^4$ is parametrized by

$$\begin{aligned} f_1(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} \cos \alpha(u) du, \\ f_2(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} \cos \alpha(u) \sin \alpha(u) du, \\ f_3(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} \sin^2 \alpha(u) du; \\ f_4(u) &= \lambda e^{-u/c}. \end{aligned} \tag{3.13}$$

4. Rotational surfaces

Consider the space $\mathbb{E}^{n+1} = \mathbb{E}^n \oplus \mathbb{E}^1$ as a subspace of $\mathbb{E}^{n+m} = \mathbb{E}^n \oplus \mathbb{E}^m$, $m \geq 2$ and Cartesian coordinates $x_1, x_2, \ldots, x_{n+m}$ and an orthonormal basis $e_1, \ldots, e_{n+m}$ in $\mathbb{E}^{n+m}$. Let $M \subset \mathbb{E}^{n+m}$ be a local surface given with the regular patch (radius vector)

$$X(u, v) = \phi(u) + f_{n+1}(u) \rho(v), \tag{4.1}$$

where

$$\phi(u) = (f_1(u), \ldots, f_n(u); 0, \ldots, 0)$$

is the vector function, such that,

$$\gamma(u) = \phi(u) + f_{n+1}(u) e_{n+1}$$

becomes a unit speed parametrization. Furthermore, the vector function

$$\rho(v) = (0, \ldots, 0; g_1(v), \ldots, g_m(v))$$

satisfying the conditions

$$\|\rho(v)\| = 1, \|\rho'(v)\| = 1 \tag{4.2}$$

specifies a curve $\rho = \rho(v)$ parametrized by a natural parameter on the unit sphere $S^{m-1} \subset \mathbb{E}^m$. Consequently, the surface M is obtained as a result of the rotation of the profile curve γ along the spherical curve ρ, which is called *rotational surface* in $\mathbb{E}^{n+m}$.

In the sequel, we consider some types of generalized rotational surfaces;

Case I. For $n = 1$ and $m = 2$, the radius vector (4.1) satisfying the indicated properties describes the *surface of revolution* in $\mathbb{E}^3$ with

$$X(u, v) = (f_1(u), f_2(u)\cos v, f_2(u)\sin v). \tag{4.3}$$

The tangent space is spanned by the vector fields

$$\begin{aligned} X_u &= ({f_1}'(u), {f_2}'(u)\cos v, {f_2}'(u)\sin v), \\ X_v &= (0, -f_2(u)\sin v, f_2(u)\cos(v)). \end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} g_{11} &= \langle X_u, X_u \rangle = 1, \\ g_{12} &= \langle X_u, X_v \rangle = 0, \\ g_{22} &= \langle X_v, X_v \rangle = (f_2(u))^2, \end{aligned}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^3$.

For a regular patch $X(u, v)$ the unit normal vector field or surface normal N is defined by

$$\begin{aligned} N(u, v) &= \frac{X_u \times X_v}{\| X_u \times X_v \|}(u, v) \\ &= ({f_2}'(u), -f_1'(u)\cos v, -f_1'(u)\sin v), \end{aligned}$$

where

$${f_1}'(u)^2 + {f_2}'(u)^2 = 1$$

and

$$\|x_u \times x_v\| = \sqrt{g_{11}g_{22} - g_{12}^2} = f_2(u) \neq 0.$$

The second partial derivatives of $X(u, v)$ are expressed as follows

$$\begin{aligned} X_{uu} &= ({f_1}''(u), {f_2}''(u)\cos v, {f_2}''(u)\sin v), \\ X_{uv} &= (0, -{f_2}'(u)\sin v, {f_2}'(u)\cos(v)), \\ X_{vv} &= (0, -f_2(u)\cos v, -f_2(u)\sin(v)). \end{aligned}$$

Similarly, the coefficients of the second fundamental form of the surface are

$$\begin{aligned} L_{11} &= \langle X_{uu}, N \rangle = -\kappa_1(u), \\ L_{12} &= \langle X_{uv}, N \rangle = 0, \\ L_{22} &= \langle X_{vv}, N \rangle = {f_1}'(u)f_2(u), \end{aligned} \tag{4.4}$$

where

$$\kappa_1(u) = f_1{}'(u) f_2{}''(u) - f_1{}''(u) f_2{}'(u) \tag{4.5}$$

is the differentiable function.

Furthermore, substituting (4.4) into (2.6)-(2.7) the Gaussian and mean curvatures of the surface become

$$K = -\frac{f_2{}''(u)}{f_2(u)} \tag{4.6}$$

and

$$2H = \frac{f_1{}'(u) - \kappa_1(u) f_2(u)}{f_2(u)}, \tag{4.7}$$

respectively (see [6]).

Assume that M is a flat surface then an easy calculation gives that the differential equation (4.6) has a non-trivial solution $f_2(u) = c_1 u + c_2$.

In [18] V. Velickovic proved the following results.

Proposition 4.1 ([18]). *Let M be a flat rotational surface given with the parametrization* (4.3). *The following assertions holds*;

i) *If the profile curve γ is of the form $\gamma(u) = (\pm u + d_1, c_2)$, then the resultant surface becomes a circular cylinder.*

ii) *If the profile curve γ is of the form $\gamma(u) = (d_1, \pm u + c_2)$, then the resultant surface becomes a portion of a plane.*

iii) *If the profile curve γ is of the form $\gamma(u) = (c_1 u, d_1 u)$, then the resultant surface becomes a circular cone.*

Proposition 4.2 ([18]). *Let M be a rotational surface in $\mathbb{E}^3$ given with the parametrization* (4.3). *If M has negative Gaussian curvature $K = \frac{-1}{c^2}$ for some constant $c > 0$, then $f_2(u) = c_1 \cosh\left(\frac{u}{c}\right) + c_2 \sinh\left(\frac{u}{c}\right)$ holds. Furthermore,*

i) *If $c_1 = -c_2 = \lambda \neq 0$, then $f_2(u) = \lambda e^{-u/c}$, and the resultant surface is called parabolic pseudo-spherical surface.*

ii) *If $c_2 = 0, c_1 = \lambda \neq 0$, then $f_2(u) = \lambda \cosh\left(\frac{u}{c}\right)$, and the resultant surface is called hyperbolic pseudo-spherical surface.*

iii) *If $c_1 = 0, c_2 = \lambda \neq 0$, then $f_2(u) = \lambda \sinh\left(\frac{u}{c}\right)$, and the resultant surface is called elliptic pseudo-spherical surface.*

Example 4.3. If we take the profile curve as the ordinary tractrix given with the parametrization (3.11), then the resultant rotational surface in $\mathbb{E}^3$ describes the *classic Beltrami surface* in $\mathbb{E}^3$ with the radius vector

$$X(u,v) = \left(\int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} du, \lambda e^{-u/c} \cos v, \lambda e^{-u/c} \sin v \right).$$

Consequently, classic Beltrami surface is a kind of parabolic pseudo-spherical surface. Eugenio Beltrami in 1868 showed that pseudo-sphere provided a model for hyperbolic geometry. Pseudo-spheres are known as surfaces with constant negative Gaussian curvature $K = \frac{-1}{c^2}$.

Case II. For $n = 2$ and $m = 2$, the radius vector (4.1) satisfying the indicated properties describes the rotational surface M in $\mathbb{E}^4$ given with the parametrization

$$X(u,v) = (f_1(u), f_2(u), f_3(u)\cos v, f_3(u)\sin v), \tag{4.8}$$

(see [7], [14]).

The tangent space is spanned by the vector fields

$$\begin{aligned} X_u &= ({f_1}'(u), {f_1}'(u), {f_3}'(u)\cos v, {f_3}'(u)\sin v), \\ X_v &= (0, 0, -f_3(u)\sin v, f_3(u)\cos(v)). \end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} g_{11} &= \langle X_u, X_u \rangle = 1, \\ g_{12} &= \langle X_u, X_v \rangle = 0, \\ g_{22} &= \langle X_v, X_v \rangle = (f_3(u))^2, \end{aligned}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$. The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{aligned} X_{uu} &= ({f_1}''(u), {f_2}''(u), {f_3}''(u)\cos v, {f_3}''(u)\sin v), \\ X_{uv} &= (0, 0, -{f_3}'(u)\sin v, {f_3}'(u)\cos(v)), \\ X_{vv} &= (0, 0, -f_3(u)\cos v, -f_3(u)\sin(v)). \end{aligned}$$

The normal space is spanned by the vector fields

(4.9)

$$\begin{aligned} N_1 &= \frac{1}{\kappa_\gamma}\left({f_1}''(u), {f_2}''(u), {f_3}''(u)\cos v, {f_3}''(u)\sin v\right), \\ N_2 &= \frac{1}{\kappa_\gamma}\left({f_2}'(u){f_3}''(u) - {f_2}''(u){f_3}'(u), {f_1}''(u){f_3}'(u) - {f_1}'(u){f_3}''(u),\right. \\ &\qquad \left.({f_1}'(u){f_2}''(u) - {f_1}''(u){f_2}'(u))\cos v, ({f_1}'(u){f_2}''(u) - {f_1}''(u){f_2}'(u))\sin v\right), \end{aligned}$$

where

$$\kappa_\gamma = \sqrt{({f_1}'')^2 + ({f_2}'')^2 + ({f_3}'')^2} \tag{4.10}$$

is the curvature of the profile curve γ.

Similarly, the coefficients of the second fundamental form of the surface are

$$\begin{aligned} L_{11}^1 &= \langle X_{uu}, N_1 \rangle = \kappa_\gamma(u), \\ L_{12}^1 &= \langle X_{uv}, N_1 \rangle = 0, \\ L_{22}^1 &= \langle X_{vv}, N_1 \rangle = -\frac{{f_3}''(u) f_3(u)}{\kappa_\gamma(u)}, \\ L_{11}^2 &= \langle X_{uu}, N_2 \rangle = 0, \\ L_{12}^2 &= \langle X_{uv}, N_2 \rangle = 0, \end{aligned} \tag{4.11}$$

$$L_{22}^2 = \langle X_{vv}, N_2 \rangle = -\frac{f_3(u)\kappa_1(u)}{\kappa_\gamma(u)},$$

where

$$\kappa_1(u) = f_1{}'(u)f_2{}''(u) - f_1{}''(u)f_2{}'(u) \tag{4.12}$$

is the curvature of the projection of the curve γ on the Oe_1e_2-plane.

Furthermore, by the use of (4.11) with (2.6)-(2.7) the Gaussian curvature and the mean curvature vector of the surface M become

$$K = -\frac{f_3{}''(u)}{f_3(u)} \tag{4.13}$$

and

$$\overrightarrow{H} = \frac{1}{2}\left\{ \left(\kappa_\gamma + \frac{K}{\kappa_\gamma} \right) N_1 - \frac{\kappa_1(u)}{f_3(u)\kappa_\gamma(u)} N_2 \right\}, \tag{4.14}$$

respectively, where κ_γ is the curvature of the profile curve γ and κ_1 is the differentiable function defined in (4.12).

Summing up the following results are proved.

Theorem 4.4. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization (4.8). Then for the Gaussian curvature K of M*

$$f_3''(u) + Kf_3(u) = 0 \tag{4.15}$$

holds.

Corollary 4.5. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization (4.8). Then M is a flat surface if and only if*

$$f_3(u) = c_1u + c_2,$$

where $c_1,\ c_2$ are real constants.

Theorem 4.6. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization (4.8). Then the mean curvature of M at point p is*

$$2H = \sqrt{\left(\kappa_\gamma + \frac{K}{\kappa_\gamma} \right)^2 + \frac{\kappa_1^2(u)}{f_3{}^2(u)\kappa_\gamma^2(u)}},$$

where κ_γ is the curvature of the profile curve γ and κ_1 is the curvature of the projection of γ on the Oe_1e_2-plane.

Proposition 4.7. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization (4.8). Then M is a minimal surface if and only if either M is a flat surface in $\mathbb{E}^4$ or a non-flat surface given with the profile curve γ*

$$f_1(u) = \frac{\lambda\sqrt{2c_2 - c_1^2}}{\sqrt{1+\lambda^2}} \ln\left(\sqrt{u^2 + 2c_1u + 2c_2} + u + c_1 \right) + c_3,$$

$$f_2(u) = \frac{\sqrt{2c_2 - c_1^2}}{\sqrt{1+\lambda^2}} \ln\left(\sqrt{u^2 + 2c_1u + 2c_2} + u + c_1 \right) + c_4, \tag{4.16}$$

$$f_3(u) = \pm\sqrt{u^2 + 2c_1u + 2c_2},$$

where c_1, c_2, c_3, c_4 *and* λ *are real constants.*

Proof. $(\Rightarrow)$: Assume that M is a minimal surface of $\mathbb{E}^4$. Then from (4.14) $\kappa_1 = 0$ and $K + \kappa_\gamma^2 = 0$. Consequently, by the use of (4.10) and the fact that γ is a unit speed curve we can get the following system of differential equations

$$\frac{f_3''(u)}{f_3(u)} = (1+\lambda^2)(f_2''(u))^2 + (f_3''(u))^2, \quad (4.17)$$
$$1 = (1+\lambda^2)(f_2'(u))^2 + (f_3'(u))^2.$$

Further, from the second differential equation in (4.17) one can get

$$f_2'(u) = \pm\frac{\sqrt{1-(f_3'(u))^2}}{\sqrt{1+\lambda^2}}. \quad (4.18)$$

Hence, substituting (4.18) into the first equation of (4.17) we obtain

$$f_3''(u)\left(1-(f_3'(u))^2 - f_3(u)f_3''(u)\right) = 0.$$

So we have two possible cases; $f_3''(u) = 0$ or $1-(f_3'(u))^2 - f_3(u)f_3''(u) = 0$. If $f_3''(u) = 0$, then $f_3(u) = au + b$ which means that M is a flat surface in $\mathbb{E}^4$. If M is a non-flat surface, then $f_3(u) = \pm\sqrt{u^2 + 2c_1u + 2c_2}$. Furthermore, substituting this value into (4.18) we get

$$f_2(u) = \frac{\sqrt{2c_2 - c_1^2}}{\sqrt{1+\lambda^2}} \ln\left(\sqrt{u^2 + 2c_1u + 2c_2} + u + c_1\right) + c_4.$$

Since, $\kappa_1 = {f_1}'(u){f_2}''(u) - {f_1}''(u){f_2}'(u) = 0$ we obtain $f_1'(u) = \lambda f_2'(u)$. Consequently, we obtain the first equation of (4.16).

$(\Leftarrow)$: Trivial. □

For any local surface $M \subset \mathbb{E}^4$ given with the regular surface patch $X(u,v)$ the normal curvature K_N is given with the following result.

Proposition 4.8 ([5])**.** *Let* $M \subset \mathbb{E}^4$ *be a local surface given with a regular patch* $X(u,v)$*. Then the normal curvature* K_N *of the surface becomes*

(4.19)
$$K_N = \frac{g_{11}(L_{12}^1L_{22}^2 - L_{12}^2L_{22}^1) - g_{12}(L_{11}^1L_{22}^2 - L_{11}^2L_{22}^1) + g_{22}(L_{11}^1L_{12}^2 - L_{11}^2L_{12}^1)}{W^3}.$$

As a consequence of (4.11) with (4.19) we get the following result.

Proposition 4.9 ([14])**.** *Any rotational surface* M *in* $\mathbb{E}^4$ *defined by* (4.8) *is a surface with flat normal connection, i.e.,* $K_N = 0$*.*

Example 4.10. If we take the profile curve as the generalized tractrix given with the parametrization (3.12), then the resultant rotational surface in $\mathbb{E}^4$ describes the generalized rotational surface in $\mathbb{E}^4$ with the parametrization

$$x_1(u,v) = \int \sqrt{1 - \frac{\lambda^2}{c^2}e^{-2u/c}} \cos\alpha(u)du,$$

$$x_2(u,v) = \int \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}} \sin \alpha(u) du, \tag{4.20}$$
$$x_3(u,v) = \lambda e^{-u/c} \cos v,$$
$$x_4(u,v) = \lambda e^{-u/c} \sin v.$$

We call such surface *generalized Beltrami surface of first kind* in $\mathbb{E}^4$.

By the use of (4.20) with (4.15) we get the following result.

Corollary 4.11. *The generalized Beltrami surface of first kind in* $\mathbb{E}^4$ *has constant Gaussian curvature* $K = -1/c^2$.

As consequence of (4.14) we obtain the following results.

Corollary 4.12. *Let M be a generalized Beltrami surface of first kind given with the parametrization* (4.20). *Then the first mean curvature of M vanishes identically if and only the angle function $\alpha(u)$ satisfies the equality*

$$\alpha'(u)^2 = \frac{1}{c^2} - \frac{(\varphi\prime)^2}{\varphi^2},$$

where φ is the differentiable function defined by

$$\varphi(u) = \sqrt{1 - \frac{\lambda^2}{c^2} e^{-2u/c}}.$$

Case III. For $n = 1$ and $m = 3$, the radius vector (4.1) satisfying the indicated properties describes the rotational surface M in $\mathbb{E}^4$ given with the parametrization

$$X(u,v) = f_1(u)\overrightarrow{e}_1 + f_2(u)\rho(v), \tag{4.21}$$

where

$$\gamma(u) = (f_1(u), f_2(u); 0, 0),$$

is the profile curve and $\rho = \rho(v)$ parametrized by

$$\rho(v) = (0; g_1(v), g_2(v), g_3(v)),$$
$$\|\rho(v)\| = 1, \|\rho'(v)\| = 1,$$

which lies on the unit sphere S^2. We know that the spherical curve ρ has the following Frenet Frames;

$$\rho'(v) = T(v),$$
$$T'(v) = \kappa_\rho(v)N(v) - \rho(v),$$
$$N'(v) = -\kappa_\rho(v)T(v).$$

Actually, the surfaces given with the parametrization (4.21) are know the *meridian surfaces* in $\mathbb{E}^4$ (see [4], [15]).

The tangent space is spanned by the vector fields

$$X_u = f_1'(u)\overrightarrow{e}_1 + f_2'(u)\rho(v),$$

$$X_v = f_2(u)\rho'(v).$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} g_{11} &= \langle X_u, X_u \rangle = 1, \\ g_{12} &= \langle X_u, X_v \rangle = 0, \\ g_{22} &= \langle X_v, X_v \rangle = {f_2}^2(u). \end{aligned}$$

The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{aligned} X_{uu} &= f_1''(u)\overrightarrow{e}_1 + f_2''(u)\rho(v), \\ X_{uv} &= f_2'(u)\rho'(v), \\ X_{vv} &= f_2(u)\rho''(v). \end{aligned}$$

The normal space of M is spanned by

$$\begin{aligned} N_1 &= N(v), \\ N_2 &= f_2'(u)\overrightarrow{e}_1 - f_1'(u)\rho(v), \end{aligned}$$

where $N(v)$ is the normal vector of the spherical curve ρ.

Similarly, the coefficients of the second fundamental form of the surface are

$$\begin{aligned} L_{11}^1 &= \langle X_{uu}, N_1 \rangle = 0, \\ L_{12}^1 &= \langle X_{uv}, N_1 \rangle = 0, \\ L_{22}^1 &= \langle X_{vv}, N_1 \rangle = \kappa_\rho(v) f_2(u), \\ L_{11}^2 &= \langle X_u, N_2 \rangle = -\kappa_\gamma(u), \\ L_{12}^2 &= \langle X_{uv}, N_2 \rangle = 0, \\ L_{22}^2 &= \langle X_{vv}, N_2 \rangle = f_1'(u) f_2(u), \end{aligned} \tag{4.22}$$

where

$$\kappa_\rho(v) = \sqrt{g_1''(v)^2 + g_2''(v)^2 + g_3''(v)^2}$$

is the curvature of the spherical curve ρ and

$$\kappa_\gamma(u) = f_1'(u) f_2''(u) - f_1''(u) f_2'(u) \tag{4.23}$$

is the curvature of profile curve γ.

Summing up the following results are proved.

Theorem 4.13. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization* (4.21)*. Then for the Gaussian curvature K of M*

$$f_2''(u) + K f_2(u) = 0 \tag{4.24}$$

holds.

Theorem 4.14. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization* (4.21)*. Then the mean curvature vector of M becomes*

$$\overrightarrow{H} = \frac{1}{2f_2(u)} \left\{ \kappa_\rho(v) N_1 + (f_1'(u) - \kappa_\gamma f_2(u)) \right\} N_2. \tag{4.25}$$

Theorem 4.15. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization* (4.21)*. Then the mean curvature of M at point p is*

$$4\left\|H\right\|^2=\frac{\kappa_\rho^2(v)+\left(f_1'(u)-\kappa_\gamma f_2(u)\right)^2}{f_2^2(u)}. \tag{4.26}$$

As a consequence of (4.22) with (4.19) we get the following result.

Proposition 4.16. *Any rotational surface in $\mathbb{E}^4$ given with the parametrization* (4.21) *has flat normal connection, i.e., $K_N=0$.*

We obtain the following results.

Corollary 4.17. *Let M be a rotational surface in $\mathbb{E}^4$ given with the parametrization* (4.21)*. Then M is a minimal surface if and only if ρ is a great circle and the profile curve γ has the parametrization*

$$\begin{aligned} f_1(u)&=\sqrt{2c_2-c_1^2}\ln\left(\sqrt{u^2+2c_1u+2c_2}+u+c_1\right)+c_3,\\ f_2(u)&=\pm\sqrt{u^2+2c_1u+2c_2}, \end{aligned} \tag{4.27}$$

where, c_1, c_2 and c_3 are real constants.

Proof. ($\Rightarrow$) Assume that M is a minimal surface of $\mathbb{E}^4$. Then from (4.26) $\kappa_\rho=0$ and $f_1'(u)-\kappa_\gamma f_2(u)=0$. Consequently, by the use of (4.23) and the fact that γ is a unit speed curve we can get

$$\left(f_2'(u)\right)^2+f_2(u)f_2''(u)=1$$

which has a nontrivial solution (4.27).

($\Leftarrow$) Trivial. □

Example 4.18. If we take the profile curve γ as the ordinary tractrix (3.11), then the resultant rotational surface M in $\mathbb{E}^4$ describes the generalized Beltrami surface $\mathbb{E}^4$ with the parametrization

$$\begin{aligned} x_1(u,v)&=\int\sqrt{1-\frac{\lambda^2}{c^2}e^{-2u/c}}du,\\ x_2(u,v)&=\lambda e^{-u/c}g_1(v),\\ x_3(u,v)&=\lambda e^{-u/c}g_2(v),\\ x_4(u,v)&=\lambda e^{-u/c}g_3(v). \end{aligned} \tag{4.28}$$

We call such surface *generalized Beltrami surface of second kind.*

By the use of (4.15) with (4.28) we get the following result.

Corollary 4.19. *The generalized Beltrami surface of first kind in $\mathbb{E}^4$ has constant Gaussian curvature $K=-1/c^2$.*

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