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Meridian Surfaces with Constant Mean Curvature in Pseudo-Euclidean 4-Space with Neutral Metric

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Abstract. In the present paper we consider a special class of Lorentz surfaces in the four-dimensional pseudo-Euclidean space with neutral metric which are one-parameter systems of meridians of rotational hypersurfaces with timelike, spacelike, or lightlike axis and call them meridian surfaces. We give the complete classification of minimal and quasi-minimal meridian surfaces. We also classify the meridian surfaces with non-zero constant mean curvature.

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1. Introduction

The study of surfaces with constant mean curvature is one of the main topics in classical differential geometry which goes back to the latter part of the eighteenth century. Lagrange was the first who found the minimal surface equation in 1761 when he looked for a necessary condition for minimizing a certain integral. Actually, the notion of mean curvature was first formally defined by Meusnier in 1776. Throughout the ninth century great mathematicians such as Gauss and Weierstrass devoted much of their studies to these surfaces. Constant mean curvature surfaces in the 3-dimensional Euclidean space are also studied intensively nowadays by many geometers for their physical interpretation. For example, surfaces with constant curvature are important mathematical models of soap films and soap bubbles.

Constant mean curvature surfaces (CMC surfaces) in arbitrary spacetime are important objects for the special role they play in the theory of general relativity. The study of CMC surfaces involves not only geometric methods but also PDE and complex analysis, which is why the theory of CMC surfaces is of great interest not only for mathematicians but also for

physicists and engineers. Surfaces with constant mean curvature in Minkowski space have been studied intensively in the past years. See for example [3,4,16,17,20].

In the four-dimensional pseudo-Euclidean space with neutral metric very few results are known on surfaces with constant mean curvature. A special case of CMC surfaces are the quasi-minimal surfaces. A Lorentzian surface in a pseudo-Riemannian manifold is called quasi-minimal (pseudo-minimal or marginally trapped) if its mean curvature vector H is lightlike at each point. Classification results on quasi-minimal surfaces in the pseudo-Euclidean space $\mathbb{E}_2^4$ have been obtained recently. The classification of quasi-minimal surfaces with parallel mean curvature vector in $\mathbb{E}_2^4$ is given in [9]. Chen [5] classified quasi-minimal Lorentz flat surfaces in $\mathbb{E}_2^4$. As an application, he gave the complete classification of biharmonic Lorentz surfaces in $\mathbb{E}_2^4$ with lightlike mean curvature vector. Several other families of quasi-minimal surfaces have also been classified. For example, quasi-minimal surfaces with constant Gauss curvature in $\mathbb{E}_2^4$ were classified in [6,11]. Quasi-minimal Lagrangian surfaces and quasi-minimal slant surfaces in complex space forms were classified, respectively, in [8,10]. For an up-to-date survey on quasi-minimal surfaces, see also [7].

In the present paper we construct special 2-dimensional Lorentz surfaces in $\mathbb{E}_2^4$ which are one-parameter systems of meridians of the rotational hypersurfaces with timelike, spacelike, or lightlike axis and call them meridian surfaces in $\mathbb{E}_2^4$. We describe all minimal meridian surfaces and show that all of them lie in hyperplanes of $\mathbb{E}_2^4$. We give the complete classification of quasi-minimal meridian surfaces (Theorems 6, 7, 8, 9). We also classify the meridian surfaces with non-zero constant mean curvature (Theorems 10, 11, 12, 13).

2. Preliminaries

Let $\mathbb{E}_2^4$ be the 4-dimensional pseudo-Euclidean space with flat metric of index 2 given in local coordinates by

$$\widetilde{g} = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2 - \mathrm{d}x_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_2^4$. We denote by $\langle ., . \rangle$ the indefinite inner scalar product associated with $\widetilde{g}$. Since $\widetilde{g}$ is an indefinite metric, a vector $v \in \mathbb{E}_2^4$ can have one of the three casual characters: it can be *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$, *timelike* if $\langle v, v \rangle < 0$, and *lightlike* if $\langle v, v \rangle = 0$ and $v \neq 0$. This terminology is inspired by general relativity and the Minkowski 4-space $\mathbb{E}_1^4$.

We use the following denotations:

$$\mathbb{S}_2^3(1) = \left\{ V \in \mathbb{E}_2^4 : \langle V, V \rangle = 1 \right\};$$
$$\mathbb{H}_1^3(-1) = \left\{ V \in \mathbb{E}_2^4 : \langle V, V \rangle = -1 \right\}.$$

The space $\mathbb{S}_2^3(1)$ is known as the de Sitter space, and the space $\mathbb{H}_1^3(-1)$ is the hyperbolic space (or the anti-de Sitter space) [18].

Given a surface M in $\mathbb{E}_2^4$, we denote by g the induced metric of $\widetilde{g}$ on M. A surface M in $\mathbb{E}_2^4$ is called *Lorentz* if the induced metric g on M is Lorentzian, i.e. at each point $p \in M$ we have the following decomposition:

$$\mathbb{E}_2^4 = T_p M \oplus N_p M$$

with the property that the restriction of the metric onto the tangent space $T_p M$ is of signature $(1, 1)$, and the restriction of the metric onto the normal space $N_p M$ is of signature $(1, 1)$.

We denote by ∇ and $\overline{\nabla}$ the Levi–Civita connections of M and $\mathbb{E}_2^4$, respectively. Then for any vector fields X, Y tangent to M the Gauss formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h is the second fundamental form of M. If D is the normal connection on the normal bundle of M, then for any normal vector field ξ and any tangent vector field X the Weingarten formula is given by

$$\overline{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where A_ξ is the shape operator with respect to ξ. The shape operator and the second fundamental form are related by the formula

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$$

for any X and Y tangent to M and any ξ normal to M.

The mean curvature vector field H of M in $\mathbb{E}_2^4$ is defined as

$$H = \frac{1}{2} \operatorname{tr} h.$$

Thus, if M is a Lorentz surface and $\{X, Y\}$ is a local orthonormal frame of the tangent bundle such that $\langle X, X \rangle = 1$, $\langle Y, Y \rangle = -1$, then the mean curvature vector field is given by the formula $H = \frac{1}{2} \left(h(X, X) - h(Y, Y) \right)$.

A surface M is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. A surface M is called *quasi-minimal* (or *pseudo-minimal*) if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H \rangle = 0$. Obviously, quasi-minimal surfaces are always non-minimal.

M is said to have constant mean curvature if $\langle H, H \rangle = const$. We shall consider Lorentz surfaces in $\mathbb{E}_2^4$ for which $\langle H, H \rangle = const \neq 0$. Such surfaces we call *CMC surfaces*.

3. Meridian Surfaces in Pseudo-Euclidean 4-Space

Meridian surfaces in Euclidean 4-space were defined first in [12] as 2-dimensional surfaces lying on a standard rotational hypersurface in $\mathbb{E}^4$. These surfaces are one-parameter systems of meridians of the rotational hypersurface; that is why they are called *meridian surfaces*. The classification of meridian surfaces with constant Gauss curvature, with constant mean curvature, Chen meridian surfaces, and meridian surfaces with parallel normal bundle is given in [12,14]. Meridian surfaces in $\mathbb{E}^4$ having pointwise 1-type Gauss map are classified in [1].

The idea from the Euclidean case is used in [13] for the construction of special families of two-dimensional spacelike surfaces lying on rotational hypersurfaces with timelike or spacelike axis in the Minkowski space $\mathbb{E}_1^4$. These surfaces are called meridian surface of elliptic or hyperbolic type, respectively. A local classification of marginally trapped meridian surfaces is given in [13]. Meridian surfaces in $\mathbb{E}_1^4$ with pointwise 1-type Gauss map are classified in [2]. The classification of meridian surfaces of elliptic or hyperbolic type with constant Gauss curvature, with constant mean curvature, Chen meridian surfaces, and meridian surfaces with parallel normal bundle is given in [15].

Following the idea from the Euclidean and Minkowski spaces, we shall construct Lorentz meridian surfaces in the pseudo-Euclidean 4-space $\mathbb{E}_2^4$ as one-parameter systems of meridians of rotational hypersurfaces with timelike or spacelike axis. We will also construct meridian surfaces which are one-parameter systems of rotational hypersurfaces with lightlike axis.

3.1. Meridian Surfaces Lying on a Rotational Hypersurface with Timelike Axis

Let $Oe_1e_2e_3e_4$ be the standard orthonormal frame in $\mathbb{E}_2^4$, i.e. $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1, \langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = -1$. First we consider a standard rotational hypersurface with timelike axis Oe_4. Similarly, we can consider a rotational hypersurface with axis Oe_3.

Since in the Minkowski space $\mathbb{E}_1^3 = \text{span} \{e_1, e_2, e_3\}$ there exist two types of spheres, namely the pseudo-sphere $\mathbb{S}_1^2(1) = \{V \in \mathbb{E}_1^3 : \langle V, V \rangle = 1\}$ and the pseudo-hyperbolic space $\mathbb{H}^2(-1) = \{V \in \mathbb{E}_1^3 : \langle V, V \rangle = -1; V^3 > 0\}$ (where V^3 is the third coordinate of V), we consider two types of rotational hypersurfaces about the axis Oe_4.

Rotational Hypersurface of First Type

Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$. The first type rotational hypersurface $\mathcal{M}^I$ in $\mathbb{E}_2^4$, obtained by the rotation of the meridian curve $m : u \rightarrow (f(u), g(u))$ about the Oe_4-axis, is parametrized as follows:

$$\mathcal{M}^I : Z(u, w^1, w^2) = f(u)(\cosh w^1 \cos w^2 \, e_1 + \cosh w^1 \sin w^2 \, e_2 + \sinh w^1 \, e_3)$$
$$+ g(u) \, e_4.$$

If we denote by $l^I(w^1, w^2) = \cosh w^1 \cos w^2 \, e_1 + \cosh w^1 \sin w^2 \, e_2 + \sinh w^1 \, e_3$ the unit position vector of the sphere $\mathbb{S}_1^2(1)$ in $\mathbb{E}_1^3$ centered at the origin O, then the parametrization of the rotational hypersurface $\mathcal{M}^I$ is written as

$$\mathcal{M}^I : Z(u, w^1, w^2) = f(u)l^I(w^1, w^2) + g(u) \, e_4.$$

Now, we shall construct Lorentz surfaces in $\mathbb{E}_2^4$ which are one-parameter systems of meridians of the hypersurface $\mathcal{M}^I$.

First we consider a smooth spacelike curve $c : l = l(v) = l^I(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ on $\mathbb{S}_1^2(1)$ parametrized by the arc-length, i.e. $\langle l', l' \rangle = 1$. We construct a two-dimensional surface $\mathcal{M}'_a$ defined by

$$\mathcal{M}'_a : z(u, v) = f(u) \, l(v) + g(u) \, e_4, \quad u \in I, \quad v \in J. \tag{1}$$

Since the surface $\mathcal{M}'_a$ is a one-parameter system of meridians of $\mathcal{M}^I$, we call it a *meridian surface on* $\mathcal{M}^I$.

The tangent space of $\mathcal{M}'_a$ is spanned by the vector fields

$$z_u = f'(u)\,l(v) + g'(u)\,e_4; \quad z_v = f(u)\,l'(v),$$

so, the coefficients of the first fundamental form of $\mathcal{M}'$ are

$$E = \langle z_u, z_u \rangle = f'^2 - g'^2; \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = f^2.$$

Since we are interested in Lorentz surfaces, we assume that the meridian curve m is timelike, i.e. $f'^2 - g'^2 < 0$. Without loss of generality we can assume that $f'^2 - g'^2 = -1$. Then the coefficients of the first fundamental form are $E = -1$; $F = 0$; $G = f^2(u)$. We consider the unit tangent vector fields $X = z_u$, $Y = \frac{z_v}{f} = l'$, which satisfy $\langle X, X \rangle = -1$, $\langle Y, Y \rangle = 1$, and $\langle X, Y \rangle = 0$.

Let $t(v) = l'(v)$ be the tangent vector field of the curve c. Since $\langle t(v), t(v) \rangle = 1$, $\langle l(v), l(v) \rangle = 1$, and $\langle t(v), l(v) \rangle = 0$, there exists a unique (up to a sign) vector field $n(v)$, such that $\langle n(v), n(v) \rangle = -1$ and $\{l(v), t(v), n(v)\}$ is an orthonormal frame field in $\mathbb{E}^3_1 = \operatorname{span}\{e_1, e_2, e_3\}$. With respect to this frame field we have the following Frenet formulas of c on $\mathbb{S}^2_1(1)$:

$$\begin{aligned} l' &= t; \\ t' &= -\kappa\,n - l; \\ n' &= -\kappa\,t, \end{aligned} \tag{2}$$

where $\kappa(v) = \langle t'(v), n(v) \rangle$ is the spherical curvature of c on $\mathbb{S}^2_1(1)$. Now we consider the following normal vector fields:

$$n_1 = n(v); \quad n_2 = g'(u)\,l(v) + f'(u)\,e_4, \tag{3}$$

which satisfy $\langle n_1, n_1 \rangle = -1$, $\langle n_2, n_2 \rangle = 1$, $\langle n_1, n_2 \rangle = 0$.

Taking into account (2) and (3) we get the following derivative formulas:

$$\begin{aligned} \overline{\nabla}_X X &= \kappa_m n_2; & \overline{\nabla}_X n_1 &= 0; \\ \overline{\nabla}_X Y &= 0; & \overline{\nabla}_Y n_1 &= -\tfrac{\kappa}{f} Y; \\ \overline{\nabla}_Y X &= \tfrac{f'}{f} Y; & \overline{\nabla}_X n_2 &= \kappa_m X; \\ \overline{\nabla}_Y Y &= \tfrac{f'}{f} X - \tfrac{\kappa}{f} n_1 - \tfrac{g'}{f} n_2; & \overline{\nabla}_Y n_2 &= \tfrac{g'}{f} Y, \end{aligned} \tag{4}$$

where κ_m denotes the curvature of the meridian curve m, which in case of a timelike curve is given by the formula $\kappa_m(u) = f''(u)g'(u) - f'(u)g''(u)$. Hence, we have

$$\begin{aligned} h(X, X) &= \kappa_m\,n_2; \\ h(X, Y) &= 0; \\ h(Y, Y) &= -\tfrac{\kappa}{f}\,n_1 - \tfrac{g'}{f}\,n_2. \end{aligned} \tag{5}$$

Formulas (5) imply that the mean curvature vector field H of $\mathcal{M}'_a$ is expressed as follows:

$$H = -\frac{\kappa}{2f}\,n_1 - \frac{f\kappa_m + g'}{2f}\,n_2.$$

Using that $g'^2 = f'^2 + 1$ and $\kappa_m = \frac{f''}{g'}$, we obtain

$$H = -\frac{\kappa}{2f}\, n_1 - \frac{f f'' + (f')^2 + 1}{2f\sqrt{f'^2 + 1}}\, n_2. \tag{6}$$

Now, let $c : l = l(v) = l^I(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ be a timelike curve on $\mathbb{S}_1^2(1)$ parametrized by the arc-length, i.e. $\langle l', l' \rangle = -1$. We consider the two-dimensional surface $\mathcal{M}_b'$ defined by

$$\mathcal{M}_b' : z(u, v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I, v \in J, \tag{7}$$

where $f'^2 - g'^2 = 1$. The surface $\mathcal{M}_b'$ is another meridian surface on $\mathcal{M}^I$.

In this case we consider an orthonormal frame field $\{l(v), t(v), n(v)\}$ of $\mathbb{E}_1^3$, such that $t = l'$, $\langle l, l \rangle = 1$, $\langle t, t \rangle = -1$, $\langle n, n \rangle = 1$. Now, the Frenet formulas of c on $\mathbb{S}_1^2(1)$ are

$$\begin{aligned} l' &= t; \\ t' &= \kappa\, n + l; \\ n' &= \kappa\, t, \end{aligned}$$

where $\kappa(v) = \langle t'(v), n(v) \rangle$. The tangent vector fields of the meridian surface $\mathcal{M}_b'$ are

$$z_u = f'\, l + g'\, e_4; \quad z_v = f\, t,$$

and since $f'^2 - g'^2 = 1$, the coefficients of the first fundamental form are $E = 1$; $F = 0$; $G = -f^2(u)$. We consider the orthonormal tangent frame $X = z_u$, $Y = \frac{z_v}{f} = t$, and the orthonormal normal frame field defined by

$$n_1 = n; \quad n_2 = g'\, l + f'\, e_4.$$

Thus, we obtain a frame field $\{X, Y, n_1, n_2\}$ of $\mathcal{M}_b'$ such that $\langle n_1, n_1 \rangle = 1, \langle n_2, n_2 \rangle = -1$ and $\langle n_1, n_2 \rangle = 0$. With respect to this frame field we have the following derivative formulas:

$$\begin{aligned} \overline{\nabla}_X X &= \kappa_m n_2; \quad \overline{\nabla}_X n_1 = 0; \\ \overline{\nabla}_X Y &= 0; \quad \overline{\nabla}_Y n_1 = \frac{\kappa}{f} Y; \\ \overline{\nabla}_Y X &= \frac{f'}{f} Y; \quad \overline{\nabla}_X n_2 = \kappa_m X; \\ \overline{\nabla}_Y Y &= \frac{f'}{f} X + \frac{\kappa}{f} n_1 - \frac{g'}{f} n_2, \quad \overline{\nabla}_Y n_2 = \frac{g'}{f} Y, \end{aligned} \tag{8}$$

where κ_m is the curvature of the meridian curve m, which in the case of a spacelike curve is given by the formula $\kappa_m = f' g'' - f'' g'$.

Equalities (8) imply that the mean curvature vector field is given by the formula

$$H = -\frac{\kappa}{2f}\, n_1 + \frac{f\kappa_m + g'}{2f}\, n_2.$$

Having in mind that $g'^2 = f'^2 - 1$ and $\kappa_m = \frac{f''}{g'}$, we obtain

$$H = -\frac{\kappa}{2f}\, n_1 + \frac{f f'' + (f')^2 - 1}{2f\sqrt{f'^2 - 1}}\, n_2. \tag{9}$$

So, we have two types of meridian surfaces lying on $\mathcal{M}^I$: meridian surfaces of type $\mathcal{M}'_a$ and meridian surfaces of type $\mathcal{M}'_b$.

Rotational Hypersurface of Second Type

Now, we consider the second type rotational hypersurface $\mathcal{M}^{II}$ in $\mathbb{E}^4_2$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the axis Oe_4, which is given by the following parametrization:

$$\mathcal{M}^{II} : Z(u, w^1, w^2) = f(u)\left(\sinh w^1 \cos w^2\, e_1 + \sinh w^1 \sin w^2\, e_2 + \cosh w^1\, e_3\right)$$
$$+ g(u)\, e_4.$$

Note that $l^{II}(w^1, w^2) = \sinh w^1 \cos w^2\, e_1 + \sinh w^1 \sin w^2\, e_2 + \cosh w^1\, e_3$ is the unit position vector of the hyperbolic sphere $\mathbb{H}^2(-1)$ in $\mathbb{E}^3_1 = \mathrm{span}\,\{e_1, e_2, e_3\}$ centered at the origin O. So, the parametrization of $\mathcal{M}^{II}$ can be written as:

$$\mathcal{M}^{II} : Z(u, w^1, w^2) = f(u)l^{II}(w^1, w^2) + g(u)\, e_4.$$

Meridian surfaces lying on the rotational hypersurface of second type $\mathcal{M}^{II}$ can be constructed as follows: let $c : l = l(v) = l^{II}(w^1(v), w^2(v))$ be a smooth curve on the hyperbolic sphere $\mathbb{H}^2(-1)$, where $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$. Then the two-dimensional surface $\mathcal{M}''$ defined by

$$\mathcal{M}'' : z(u, v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\ v \in J \qquad (10)$$

is a one-parameter system of meridians of $\mathcal{M}^{II}$, which we call a *meridian surface on* $\mathcal{M}^{II}$.

The tangent space of $\mathcal{M}''$ is spanned by the vector fields

$$z_u = f'(u)\, l(v) + g'(u)\, e_4; \quad z_v = f(u)\, l'(v),$$

so the coefficients of the first fundamental form of $\mathcal{M}''$ are

$$E = \langle z_u, z_u \rangle = -(f'^2 + g'^2); \quad F = \langle z_u, z_v \rangle = 0; \quad G = \langle z_v, z_v \rangle = f^2 \langle l', l' \rangle.$$

Since c is a curve lying on $\mathbb{H}^2(-1)$, we have $\langle l, l \rangle = -1$, which implies that the tangent vector field $t = l'$ satisfies $\langle t, t \rangle = 1$. Without loss of generality we suppose that $f'^2 + g'^2 = 1$. Then the coefficients of the first fundamental form of $\mathcal{M}''$ are $E = -1$; $F = 0$; $G = f^2$.

We consider an orthonormal frame field $\{l(v), t(v), n(v)\}$ of c satisfying the conditions $\langle l, l \rangle = -1$, $\langle t, t \rangle = 1$, $\langle n, n \rangle = 1$. The Frenet formulas of c on $\mathbb{H}^2(-1)$ are

$$\begin{aligned} l' &= t; \\ t' &= \kappa\, n + l; \\ n' &= -\kappa\, t, \end{aligned} \qquad (11)$$

where $\kappa(v) = \langle t'(v), n(v) \rangle$ is the curvature of c on $\mathbb{H}^2(-1)$.

Let us consider the following orthonormal frame field of $\mathcal{M}''$:

$$X = z_u; \quad Y = \frac{z_v}{f} = t; \quad n_1 = n(v); \quad n_2 = -g'(u)\, l(v) + f'(u)\, e_4.$$

This frame field satisfies $\langle X, X \rangle = -1$, $\langle Y, Y \rangle = 1$, $\langle X, Y \rangle = 0$, $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$, $\langle n_1, n_2 \rangle = 0$. Taking into account (11) we get the following derivative formulas:

$$\overline{\nabla}_X X = \kappa_m n_2; \quad \overline{\nabla}_X n_1 = 0;$$
$$\overline{\nabla}_X Y = 0; \quad \overline{\nabla}_Y n_1 = -\frac{\kappa}{f} Y;$$
$$\overline{\nabla}_Y X = \frac{f'}{f} Y; \quad \overline{\nabla}_X n_2 = -\kappa_m X;$$
$$\overline{\nabla}_Y Y = \frac{f'}{f} X + \frac{\kappa}{f} n_1 - \frac{g'}{f} n_2; \quad \overline{\nabla}_Y n_2 = -\frac{g'}{f} Y,$$

where $\kappa_m = f'g'' - f''g'$. Hence, we obtain the formulas

$$h(X, X) = \kappa_m n_2;$$
$$h(X, Y) = 0;$$
$$h(Y, Y) = \frac{\kappa}{f} n_1 - \frac{g'}{f} n_2,$$

which imply that the normal mean curvature vector field of $\mathcal{M}''$ is given by

$$H = \frac{\kappa}{2f} n_1 + \frac{ff'' + (f')^2 - 1}{2f\sqrt{1 - f'^2}} n_2.$$

Note that we can construct only one type of Lorentz meridian surfaces lying on the rotational hypersurface $\mathcal{M}^{II}$.

3.2. Meridian Surfaces Lying on a Rotational Hypersurface with Spacelike Axis

In a similar way, we can construct meridian surfaces lying on a rotational hypersurface with spacelike axis Oe_1 (or Oe_2).

In the Minkowski space $\mathbb{E}_2^3 = \text{span}\{e_2, e_3, e_4\}$ there exist two types of spheres, namely the de Sitter space $\mathbb{S}_2^2(1) = \{V \in \mathbb{E}_2^3 : \langle V, V \rangle = 1\}$, and the hyperbolic space $\mathbb{H}_1^2(-1) = \{V \in \mathbb{E}_2^3 : \langle V, V \rangle = -1\}$. So, we can consider two types of rotational hypersurfaces about the axis Oe_1.

Rotational Hypersurface of First Type

Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$. We denote by $\tilde{l}^I(w^1, w^2) = \cosh w^1 e_2 + \sinh w^1 \cos w^2 e_3 + \sinh w^1 \sin w^2 e_4$ the unit position vector of the sphere $\mathbb{S}_2^2(1)$ in $\mathbb{E}_2^3 = \text{span}\{e_2, e_3, e_4\}$ centered at the origin O. Then, the first type rotational hypersurface $\widetilde{\mathcal{M}}^I$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the axis Oe_1, is parametrized by

$$\widetilde{\mathcal{M}}^I : Z(u, w^1, w^2) = g(u) e_1 + f(u)(\cosh w^1 e_2 + \sinh w^1 \cos w^2 e_3$$
$$+ \sinh w^1 \sin w^2 e_4),$$

or equivalently,

$$\widetilde{\mathcal{M}}^I : Z(u, w^1, w^2) = g(u) e_1 + f(u) \tilde{l}^I(w^1, w^2).$$

Lorentz meridian surfaces lying on $\widetilde{\mathcal{M}}^I$ are one-parameter systems of meridians of $\widetilde{\mathcal{M}}^I$. They can be constructed as follows: let $c : l = l(v) = \tilde{l}^I(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ be a smooth curve on $\mathbb{S}_2^2(1)$. We consider the two-dimensional surface $\widetilde{\mathcal{M}}'$ defined by

$$\widetilde{\mathcal{M}}' : z(u, v) = g(u) e_1 + f(u) l(v), \quad u \in I, \quad v \in J.$$

It is a one-parameter system of meridians of $\widetilde{\mathcal{M}}^I$, so we call $\widetilde{\mathcal{M}}'$ a *meridian surface on* $\widetilde{\mathcal{M}}^I$.

It can easily be seen that the meridian surface $\mathcal{M}''$, defined by (10), can be transformed into the surface $\widetilde{\mathcal{M}}'$ by the transformation T given by the matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{12}$$

So, the meridian surface $\mathcal{M}''$ lying on $\mathcal{M}^{II}$ and the meridian surface $\widetilde{\mathcal{M}}'$ lying on $\widetilde{\mathcal{M}}^I$ are congruent. Hence, all results concerning the surface $\mathcal{M}''$ hold true for the surface $\widetilde{\mathcal{M}}'$.

Rotational Hypersurface of Second Type

The second type rotational hypersurface $\widetilde{\mathcal{M}}^{II}$, obtained by the rotation of the meridian curve m about Oe_1, is given by the following parametrization:

$$\widetilde{\mathcal{M}}^{II} : Z(u, w^1, w^2) = g(u)\,e_1 + f(u)(\sinh w^1\,e_2 + \cosh w^1 \cos w^2\,e_3$$
$$+ \cosh w^1 \sin w^2\,e_4).$$

Here, $\tilde{l}^{II}(w^1, w^2) = \sinh w^1\,e_2 + \cosh w^1 \cos w^2\,e_3 + \cosh w^1 \sin w^2\,e_4$ is the unit position vector of the hyperbolic sphere $\mathbb{H}^2_1(-1)$ in $\mathbb{E}^3_2 = \text{span}\,\{e_2, e_3, e_4\}$ centered at the origin O. So, the parametrization of $\widetilde{\mathcal{M}}^{II}$ can be written as

$$\widetilde{\mathcal{M}}^{II} : Z(u, w^1, w^2) = g(u)\,e_1 + f(u)\,\tilde{l}^{II}(w^1, w^2).$$

We can construct two types of meridian surfaces lying on the second type rotational hypersurface $\widetilde{\mathcal{M}}^{II}$.

First, we consider a smooth spacelike curve $c : l = l(v) = \tilde{l}^{II}(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ lying on the hyperbolic sphere $\mathbb{H}^2_1(-1)$ in $\mathbb{E}^3_2$. We assume that c is parametrized by the arc-length, i.e. $\langle l', l' \rangle = 1$. Let $\widetilde{\mathcal{M}}''_a$ be the surface lying on $\widetilde{\mathcal{M}}^{II}$ and defined by

$$\widetilde{\mathcal{M}}''_a : z(u, v) = g(u)\,e_1 + f(u)\,l(v), \quad u \in I, v \in J.$$

The tangent space of $\widetilde{\mathcal{M}}''_a$ is spanned by the vector fields

$$z_u = g'(u)\,e_1 + f'(u)\,l(v); \qquad z_v = f(u)\,l'(v),$$

so, the coefficients of the first fundamental form are

$$E = g'^2 - f'^2; \quad F = 0; \quad G = f^2.$$

Since we are interested in Lorentz surfaces, we assume that $f'^2 - g'^2 = 1$. Then the coefficients of the first fundamental form are $E = -1$; $F = 0$; $G = f^2$. It is easy to see that under the transformation T given by (12) the surface $\mathcal{M}'_b$ is transformed into the surface $\widetilde{\mathcal{M}}''_a$. So, all results concerning the surface $\mathcal{M}'_b$ hold true for the surface $\widetilde{\mathcal{M}}''_a$.

Second, we consider a timelike curve $c : l = l(v) = \tilde{l}^{II}(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ lying on the hyperbolic sphere $\mathbb{H}^2_1(-1)$ and parametrized by the

arc-length, i.e. $\langle l', l' \rangle = -1$. Then the surface $\widetilde{\mathcal{M}}_b''$ defined by

$$\widetilde{\mathcal{M}}_b'' : z(u, v) = g(u)\,e_1 + f(u)\,l(v), \quad u \in I,\ v \in J,$$

where $f'^2 - g'^2 = -1$ is a Lorentz meridian surface lying on $\widetilde{\mathcal{M}}^{II}$. It is clear that the meridian surfaces $\mathcal{M}_a'$ and $\widetilde{\mathcal{M}}_b''$ are congruent up to the transformation T.

3.3. Meridian Surfaces Lying on a Rotational Hypersurface with Lightlike Axis

In this subsection we will construct meridian surfaces lying on a rotational hypersurface with lightlike axis. For convenience we denote $\xi_1 = \dfrac{e_2 + e_4}{\sqrt{2}}$, $\xi_2 = \dfrac{-e_2 + e_4}{\sqrt{2}}$ and consider the pseudo-orthonormal base $\{e_1, e_3, \xi_1, \xi_2\}$ of $\mathbb{E}_2^4$. Note that $\langle \xi_1, \xi_1 \rangle = 0$, $\langle \xi_2, \xi_2 \rangle = 0$, $\langle \xi_1, \xi_2 \rangle = -1$. A rotational hypersurface with lightlike axis in $\mathbb{E}_2^4$ can be parametrized by

$$\mathcal{M}^{III} : Z(u, w^1, w^2) = f(u)\,w^1 \cosh w^2\, e_1 + f(u)\,w^1 \sinh w^2\, e_3$$
$$+ \left(f(u)\frac{(w^1)^2}{2} + g(u) \right) \xi_1 + f(u)\xi_2,$$

where $f = f(u)$, $g = g(u)$ are smooth functions, defined in an interval $I \subset \mathbb{R}$ and $f(u) > 0$, $u \in I$.

Let $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$ and consider the surface $\mathcal{M}'''$ in $\mathbb{E}_2^4$ given by

$$\mathcal{M}''' : z(u, v) = Z(u, w^1(v), w^2(v)), \tag{13}$$

where $u \in I$, $v \in J$. The surface $\mathcal{M}'''$, defined by (13), is a one-parameter system of meridians of the rotational hypersurface $\mathcal{M}^{III}$. So, we call $\mathcal{M}'''$ a *meridian surface on* $\mathcal{M}^{III}$.

Without loss of generality we can assume that $w^1 = \rho(v)$, $w^2 = v$. Then the surface $\mathcal{M}'''$ is parametrized as follows:

$$\mathcal{M}''' : z(u, v) = f(u) \left(\rho(v) \cosh v\, e_1 + \rho(v) \sinh v\, e_3 + \frac{\rho^2(v)}{2}\,\xi_1 + \xi_2 \right) + g(u)\,\xi_1. \tag{14}$$

If we denote $l(v) = \rho(v) \cosh v\, e_1 + \rho(v) \sinh v\, e_3 + \frac{\rho^2(v)}{2}\,\xi_1 + \xi_2$, then the parametrization (14) is written as

$$\mathcal{M}''' : z(u, v) = f(u)\,l(v) + g(u)\,\xi_1.$$

The tangent vector fields of $\mathcal{M}'''$ are

$$z_u = f'\rho \cosh v\, e_1 + f'\rho \sinh v\, e_3 + \left(f'\frac{\rho^2}{2} + g' \right) \xi_1 + f'\,\xi_2;$$
$$z_v = f(\dot{\rho} \cosh v + \rho \sinh v)\, e_1 + f(\dot{\rho} \sinh v + \rho \cosh v)\, e_3 + f\rho\dot{\rho}\,\xi_1, \tag{15}$$

where $\dot{\rho}$ denotes the derivative of ρ with respect to v. So, the coefficients of the first fundamental form of $\mathcal{M}'''$ are

$$E = -2f'(u)g'(u); \quad F = 0; \quad G = f^2(u)(\dot{\rho}^2(v) - \rho^2(v)).$$

Since we are studying Lorentz surfaces, in the case $\dot{\rho}^2(v) - \rho^2(v) > 0$ we assume that $f'(u)g'(u) > 0$; in the case $\dot{\rho}^2(v) - \rho^2(v) < 0$ we assume that $f'(u)g'(u) < 0$.

We consider the tangent frame field defined by $X = \dfrac{z_u}{\sqrt{2\varepsilon f'g'}}$, $Y = \dfrac{z_v}{f\sqrt{\varepsilon(\dot{\rho}^2 - \rho^2)}}$, where $\varepsilon = 1$ in the case $\dot{\rho}^2 - \rho^2 > 0$, $f'g' > 0$, and $\varepsilon = -1$ in the case $\dot{\rho}^2 - \rho^2 < 0$, $f'g' < 0$. So, we have $\langle X, X \rangle = -\varepsilon$, $\langle Y, Y \rangle = \varepsilon$, $\langle X, Y \rangle = 0$. We choose the following normal frame field:

$$
n_1 = \sqrt{\frac{\varepsilon f'}{2g'}}(\rho \cosh v \, e_1 + \rho \sinh v \, e_3 + \frac{f'\rho^2 - 2g'}{2f'}\xi_1 + \xi_2);
$$

$$
n_2 = \frac{1}{\sqrt{\varepsilon(\dot{\rho}^2 - \rho^2)}}((\dot{\rho}\sinh v + \rho \cosh v)\, e_1 + (\dot{\rho}\cosh v + \rho \sinh v)\, e_3 + \rho^2 \, \xi_1),
$$

$$(16)$$

which satisfies $\langle n_1, n_1 \rangle = \varepsilon$, $\langle n_2, n_2 \rangle = -\varepsilon$, $\langle n_1, n_2 \rangle = 0$.

Taking into account (15), we calculate the second partial derivatives of $z(u, v)$:

$$
z_{uu} = f''\rho \cosh v \, e_1 + f''\rho \sinh v \, e_3 + \left(f''\frac{\rho^2}{2} + g''\right)\xi_1 + f''\xi_2;
$$

$$
z_{uv} = f'(\dot{\rho}\cosh v + \rho \sinh v)\, e_1 + f'(\dot{\rho}\sinh v + \rho \cosh v)\, e_3 + f'\rho\dot{\rho}\,\xi_1;
$$

$$
z_{vv} = f\,((\ddot{\rho} + \rho)\cosh v + 2\dot{\rho}\sinh v)\, e_1 + f\,((\ddot{\rho} + \rho)\sinh v + 2\dot{\rho}\cosh v)\, e_3
$$
$$
+ f\,(\dot{\rho}^2 + \rho\ddot{\rho})\,\xi_1.
$$

The last equalities together with (16) imply

$$
\langle z_{uu}, n_1 \rangle = \frac{f''g' - g''f'}{\sqrt{2\varepsilon f'g'}}; \quad \langle z_{uu}, n_2 \rangle = 0;
$$

$$
\langle z_{uv}, n_1 \rangle = 0; \quad \langle z_{uv}, n_2 \rangle = 0;
$$

$$
\langle z_{vv}, n_1 \rangle = -f\sqrt{\frac{\varepsilon f'}{2g'}}(\dot{\rho}^2 - \rho^2); \quad \langle z_{vv}, n_2 \rangle = f\frac{\rho\ddot{\rho} - 2\dot{\rho}^2 + \rho^2}{\sqrt{\varepsilon(\dot{\rho}^2 - \rho^2)}}.
$$

Hence, we obtain

$$
h(X, X) = \varepsilon\frac{f''g' - g''f'}{(2\varepsilon f'g')^{\frac{3}{2}}}\, n_1;
$$

$$
h(X, Y) = 0; \tag{17}
$$

$$
h(Y, Y) = -\frac{1}{f}\sqrt{\frac{\varepsilon f'}{2g'}}\, n_1 - \varepsilon\frac{\rho\ddot{\rho} - 2\dot{\rho}^2 + \rho^2}{f(\varepsilon(\dot{\rho}^2 - \rho^2))^{\frac{3}{2}}}\, n_2.
$$

Note that $\dfrac{\varepsilon(f''g' - g''f')}{(2\varepsilon f'g')^{\frac{3}{2}}}$ is the curvature of the meridian curves (the parametric u-lines) of $\mathcal{M}'''$. We denote $\kappa_m(u) = \dfrac{\varepsilon(f''g' - g''f')}{(2\varepsilon f'g')^{\frac{3}{2}}}$. The function $\kappa(v) = \dfrac{\rho\ddot{\rho} - 2\dot{\rho}^2 + \rho^2}{(\varepsilon(\dot{\rho}^2 - \rho^2))^{\frac{3}{2}}}$ is the curvature of the curve $c : l = l(v) = \rho(v)\cosh$

$v\,e_1 + \rho(v)\sinh v\,e_3 + \frac{\rho^2(v)}{2}\,\xi_1 + \xi_2$. So, formulas (17) take the form

$$
\begin{aligned}
h(X, X) &= \kappa_m\,n_1; \\
h(X, Y) &= 0; \\
h(Y, Y) &= -\frac{1}{f}\sqrt{\frac{\varepsilon f'}{2g'}}\,n_1 - \varepsilon\frac{\kappa}{f}\,n_2.
\end{aligned}
\tag{18}
$$

It follows from (18) that the mean curvature vector field H of the meridian surface $\mathcal{M}'''$ is expressed as

$$
H = -\frac{\varepsilon}{2}\left(\kappa_m + \frac{1}{f}\sqrt{\frac{\varepsilon f'}{2g'}}\right)n_1 - \frac{\kappa}{2f}\,n_2.
$$

Without loss of generality we can assume that $2\varepsilon f'g' = 1$, which implies $\kappa_m = \frac{f''}{f'}$. Hence,

$$
H = -\frac{\varepsilon(f f'' + (f')^2)}{2f f'}\,n_1 - \frac{\kappa}{2f}\,n_2.
\tag{19}
$$

Similarly to the construction of the meridian surface $\mathcal{M}'''$, one can construct other meridian surfaces lying on rotational hypersurfaces with lightlike axis which are congruent to the surface given by (14).

In the following sections we will study four types of Lorentz meridian surfaces in $\mathbb{E}_2^4$, namely the surfaces denoted by $\mathcal{M}'_a$, $\mathcal{M}'_b$, $\mathcal{M}''$, and $\mathcal{M}'''$.

4. Minimal Meridian Surfaces in $\mathbb{E}_2^4$

In this section we give the classification of all minimal meridian surfaces in $\mathbb{E}_2^4$.

Theorem 1. *Let $\mathcal{M}'_a$ be a meridian surface on $\mathcal{M}^I$ defined by (1). Then $\mathcal{M}'_a$ is minimal if and only if the curve c has zero spherical curvature and the meridian curve m is given by*

$$
f(u) = \pm\sqrt{-u^2 + 2au + b}, \quad g(u) = \pm\sqrt{a^2 + b}\,\arcsin\frac{u - a}{\sqrt{a^2 + b}} + c,
$$

where $a = const$, $b = const$, $c = const$.

Proof. The mean curvature vector field H of the meridian surface $\mathcal{M}'_a$ is given by formula (6). Hence, $\mathcal{M}'_a$ is minimal if and only if the curvature of c is $\kappa = 0$ and the function $f(u)$ satisfies the following equation:

$$
f f'' + (f')^2 + 1 = 0.
$$

The solutions of this differential equation are given by the formula $f(u) = \pm\sqrt{-u^2 + 2au + b}$, where $a = const$, $b = const$. Having in mind that $g' = \sqrt{f'^2 + 1}$, we get the following equation for $g(u)$:

$$
g' = \pm\frac{\sqrt{a^2 + b}}{\sqrt{-u^2 + 2au + b}}.
$$

Integrating the above equation we obtain

$$g(u) = \pm\sqrt{a^2 + b}\arcsin\frac{u - a}{\sqrt{a^2 + b}} + c, \quad c = const.$$

□

Note that if $\mathcal{M}'_a$ is minimal, then $\kappa = 0$ and from (4) we get that $\overline{\nabla}_X n_1 = 0$, $\overline{\nabla}_Y n_1 = 0$. This means that the normal vector field n_1 is constant. Hence, the surface $\mathcal{M}'_a$ lies in the constant 3-dimensional space $\mathbb{E}^3_1 = \text{span}\{X, Y, n_2\}$. Consequently, $\mathcal{M}'_a$ lies in a hyperplane of $\mathbb{E}^4_2$.

Theorem 2. *Let $\mathcal{M}'_b$ be a meridian surface on $\mathcal{M}^I$ defined by (7). Then $\mathcal{M}'_b$ is minimal if and only if the curve c has zero spherical curvature and the meridian curve m is given by*

$$f(u) = \pm\sqrt{u^2 + 2au + b}, \quad g(u) = \pm\sqrt{a^2 - b}\ln|u + a + \sqrt{u^2 + 2au + b}| + c,$$

where $a = const$, $b = const$, $c = const$, $a^2 - b > 0$.

Proof. Using that the mean curvature vector field H of $\mathcal{M}'_b$ is given by formula (9), we get that $\mathcal{M}'_b$ is minimal if and only if $\kappa = 0$ and the function $f(u)$ satisfies the equation

$$ff'' + (f')^2 - 1 = 0.$$

The solutions of this differential equation are given by the formula $f(u) = \pm\sqrt{u^2 + 2au + b}$, where $a = const$, $b = const$. Using that $g' = \sqrt{f'^2 - 1}$, we get the following equation for $g(u)$:

$$g' = \pm\frac{\sqrt{a^2 - b}}{\sqrt{u^2 + 2au + b}}.$$

Integrating the last equation we obtain

$$g(u) = \pm\sqrt{a^2 - b}\ln|u + a + \sqrt{u^2 + 2au + b}| + c, \quad c = const.$$

□

Note that if $\mathcal{M}'_b$ is minimal, we have $\kappa = 0$ and again we obtain $\overline{\nabla}_X n_1 = 0$, $\overline{\nabla}_Y n_1 = 0$, i.e. the normal vector field n_1 is constant. In this case the surface $\mathcal{M}'_b$ lies in the constant 3-dimensional space $\mathbb{E}^3_2 = \text{span}\{X, Y, n_2\}$. Consequently, $\mathcal{M}'_b$ lies in a hyperplane of $\mathbb{E}^4_2$.

Theorem 3. *Let $\mathcal{M}''$ be a meridian surface on $\mathcal{M}^{II}$ defined by (10). Then $\mathcal{M}''$ is minimal if and only if the curve c has zero spherical curvature and the meridian curve m is given by*

$$f(u) = \pm\sqrt{u^2 + 2au + b}, \quad g(u) = \pm\sqrt{b - a^2}\ln|u + a + \sqrt{u^2 + 2au + b}| + c,$$

where $a = const$, $b = const$, $c = const$, $b - a^2 > 0$.

The proof is similar to the proof of the previous two theorems. Again we have that if $\mathcal{M}''$ is minimal, then it lies in a constant 3-dimensional hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$.

Theorem 4. *Let $\mathcal{M}'''$ be a meridian surface on $\mathcal{M}^{III}$ defined by (14). Then $\mathcal{M}'''$ is minimal if and only if the curve c has zero curvature and the meridian curve m is given by*

$$f(u) = \sqrt{au + b}, \quad g(u) = \frac{2}{3a^2}(au + b)^{\frac{3}{2}} + c,$$

where $a = const \neq 0$, $b = const$, $c = const$.

The proof is similar to the proofs of the previous theorems. We also obtain that if $\mathcal{M}'''$ is minimal, then $\overline{\nabla}_X n_2 = 0$, $\overline{\nabla}_Y n_2 = 0$, i.e. the normal vector field n_2 is constant. In this case the surface $\mathcal{M}'''$ lies in a constant 3-dimensional space orthogonal to n_2, i.e. $\mathcal{M}'''$ lies in a hyperplane of $\mathbb{E}_2^4$.

Finally, we can formulate the following:

Corollary 5. *There are no minimal meridian surfaces lying fully in $\mathbb{E}_2^4$.*

5. Quasi-Minimal Meridian Surfaces in $\mathbb{E}_2^4$

In this section we classify all quasi-minimal meridian surfaces in the pseudo-Euclidean 4-space $\mathbb{E}_2^4$.

Theorem 6. *Let $\mathcal{M}_a'$ be a meridian surface on $\mathcal{M}^I$ defined by (1). Then $\mathcal{M}_a'$ is quasi-minimal if and only if the curve c has constant curvature $\kappa = a = const$, $a \neq 0$ and the meridian curve m is determined by $f' = \varphi(f)$, where*

$$\varphi(t) = \pm\frac{1}{t}\sqrt{(\pm at + c)^2 - t^2}, \quad c = const,$$

$g(u)$ is defined by $g' = \sqrt{f'^2 + 1}$.

Proof. Using formula (6) for the mean curvature vector field H of the meridian surface $\mathcal{M}_a'$, we get that $\mathcal{M}_a'$ is quasi-minimal, i.e. $H \neq 0$ and $\langle H, H \rangle = 0$, if and only if

$$\frac{\left(ff'' + (f')^2 + 1\right)^2}{f'^2 + 1} = \kappa^2, \quad \kappa \neq 0.$$

The left-hand side of this equation is a function of u, and the right-hand side of the equation is a function of v. Hence, we obtain

$$\kappa = a, \quad a = const \neq 0;$$
$$\left(ff'' + (f')^2 + 1\right)^2 = a^2(f'^2 + 1).$$

So, the meridian m is determined by the following differential equation:

$$ff'' + (f')^2 + 1 = \pm a\sqrt{f'^2 + 1}. \tag{20}$$

The solutions of the above differential equation can be found in the following way. If we set $f' = \varphi(f)$ in Eq. (20), we obtain that the function $\varphi = \varphi(t)$ is a solution of the equation:

$$\frac{t}{2}(\varphi^2)' + \varphi^2 + 1 = \pm a\sqrt{\varphi^2 + 1}. \tag{21}$$

Now, we set $z(t) = \sqrt{\varphi^2(t) + 1}$, so, Eq. (21) takes the form

$$z' + \frac{1}{t} z = \pm \frac{a}{t}.$$

The general solution of the last equation is given by the formula $z(t) = \frac{\pm at + c}{t}$, $c = const$. Hence, the general solution of (21) is

$$\varphi(t) = \pm \frac{1}{t} \sqrt{(\pm at + c)^2 - t^2}.$$

$\square$

Similarly to the proof of Theorem 6 we obtain the classification of all quasi-minimal meridian surfaces of type $\mathcal{M}'_b$:

Theorem 7. *Let $\mathcal{M}'_b$ be a meridian surface on $\mathcal{M}^I$ defined by (7). Then $\mathcal{M}'_b$ is quasi-minimal if and only if the curve c has constant curvature $\kappa = a = const$, $a \neq 0$ and the meridian curve m is determined by $f' = \varphi(f)$ where*

$$\varphi(t) = \pm \frac{1}{t} \sqrt{(c \pm at)^2 + t^2}, \quad c = const,$$

$g(u)$ is defined by $g' = \sqrt{f'^2 - 1}$.

The classification of all quasi-minimal meridian surface of type $\mathcal{M}''$ is given in the next theorem.

Theorem 8. *Let $\mathcal{M}''$ be a meridian surface on $\mathcal{M}^{II}$ defined by (10). Then $\mathcal{M}''$ is quasi-minimal if and only if the curve c has constant curvature $\kappa = a = const$, $a \neq 0$ and the meridian curve m is determined by $f' = \varphi(f)$ where*

$$\varphi(t) = \pm \frac{1}{t} \sqrt{t^2 - (c \pm at)^2}, \quad c = const,$$

$g(u)$ is defined by $g' = \sqrt{1 - f'^2}$.

The proof is similar to the proof of Theorem 6.

Theorem 9. *Let $\mathcal{M}'''$ be a meridian surface on $\mathcal{M}^{III}$ defined by (14). Then $\mathcal{M}'''$ is quasi-minimal if and only if the curve c has constant curvature $\kappa = a = const$, $a \neq 0$ and the meridian curve m is determined by $f' = \varphi(f)$ where*

$$\varphi(t) = \frac{\pm at + c}{t}, \quad c = const,$$

$g(u)$ is defined by $g' = \frac{1}{2\varepsilon f'}$.

Proof. Using formula (19) for the mean curvature vector field H of the meridian surface $\mathcal{M}'''$, we get that $\mathcal{M}'''$ is quasi-minimal if and only if

$$\left(f f'' + (f')^2 \right)^2 = \kappa^2 (f')^2, \quad \kappa \neq 0.$$

The last equality implies that

$$\kappa = a, \quad a = const \neq 0;$$
$$f f'' + (f')^2 = \pm a f'.$$

The solutions of the above differential equation can be found in the following way: setting $f' = \varphi(f)$, we obtain that the function $\varphi = \varphi(t)$ is a solution of the equation

$$\varphi' + \frac{1}{t}\varphi = \pm\frac{a}{t}.$$

The general solution of the last equation is given by the formula $\varphi(t) = \frac{\pm at + c}{t}$, $c = const$. $\qquad\square$

6. Meridian Surfaces with Constant Mean Curvature in $\mathbb{E}^4_2$

In this section we shall classify all meridian surfaces with non-zero constant mean curvature.

Let $\mathcal{M}'_a$ be a meridian surface on $\mathcal{M}^I$ defined by (1). The mean curvature vector field H of $\mathcal{M}'_a$ is given by formula (6). So, we have

$$\langle H, H \rangle = \frac{\left(ff'' + (f')^2 + 1\right)^2 - \kappa^2(f'^2 + 1)}{4f^2(f'^2 + 1)}. \tag{22}$$

In the next theorem we classify the meridian surfaces of type $\mathcal{M}'_a$ for which $\langle H, H \rangle = c = const, \ c \neq 0$.

Theorem 10. *Let $\mathcal{M}'_a$ be a meridian surface on $\mathcal{M}^I$ defined by (1). Then $\mathcal{M}'_a$ has non-zero constant mean curvature, i.e. $\langle H, H \rangle = c = const, c \neq 0$, if and only if the curve c has constant curvature $\kappa = a = const, a \neq 0$ and the meridian curve m is determined by $f' = \varphi(f)$ where*

$$\varphi(t) = \pm\frac{1}{t}\sqrt{\left(b \pm \frac{t}{2}\sqrt{a^2 + 4ct^2} \pm \frac{a^2}{4\sqrt{c}}\ln|2\sqrt{c}t + \sqrt{a^2 + 4ct^2}|\right)^2 - t^2}, \quad \text{if } c > 0,$$

$$\varphi(t) = \pm\frac{1}{t}\sqrt{\left(b \pm \frac{t}{2}\sqrt{a^2 - 4ct^2} \pm \frac{a^2}{4\sqrt{-c}}\arcsin\frac{2\sqrt{-c}t}{a}\right)^2 - t^2}, \quad \text{if } c < 0,$$

$b = const$, and $g(u)$ is defined by $g' = \sqrt{f'^2 + 1}$.

Proof. Let $\mathcal{M}'_a$ be a meridian surface on $\mathcal{M}^I$. It follows from (22) that the condition on $\mathcal{M}'_a$ to have non-zero constant mean curvature, i.e. $\langle H, H \rangle = c = const, c \neq 0$ is equivalent to the equality

$$\frac{\left(ff'' + (f')^2 + 1\right)^2 - \kappa^2(f'^2 + 1)}{4f^2(f'^2 + 1)} = c.$$

The last equality can be written as

$$\frac{\left(ff'' + (f')^2 + 1\right)^2 - 4cf^2(f'^2 + 1)}{f'^2 + 1} = \kappa^2. \tag{23}$$

Since the left-hand side of (23) is a function of u, and the right-hand side of (23) is a function of v, we obtain

$$\kappa = a, \quad a = const \neq 0;$$
$$\frac{\left(ff'' + (f')^2 + 1\right)^2 - 4cf^2(f'^2 + 1)}{f'^2 + 1} = a^2.$$

Hence, the meridian m is determined by the solutions of the following differential equation:

$$\left(ff'' + (f')^2 + 1\right)^2 - 4cf^2(f'^2 + 1) = a^2(f'^2 + 1).$$

We can find the solutions of the last equation by setting $f' = \varphi(f)$. Then we obtain that the function $\varphi = \varphi(t)$ is a solution of the following equation:

$$\left(\frac{t}{2}(\varphi^2)' + \varphi^2 + 1\right)^2 - 4ct^2(\varphi^2 + 1) = a^2(\varphi^2 + 1). \tag{24}$$

If we set $z(t) = \sqrt{\varphi^2(t) + 1}$, from equation (24) we get

$$z' + \frac{1}{t}z = \pm\frac{\sqrt{a^2 + 4ct^2}}{t}.$$

The general solution of the last equation is given by the formula $z(t) = \frac{1}{t}\left(b \pm \int \sqrt{a^2 + 4ct^2}\,dt\right)$, where $b = const$.

If $c > 0$, i.e. the mean curvature vector field H is spacelike, then $\int \sqrt{a^2 + 4ct^2}\,dt = \frac{t}{2}\sqrt{a^2 + 4ct^2} + \frac{a^2}{4\sqrt{c}}\ln|2\sqrt{c}t + \sqrt{a^2 + 4ct^2}|$. Hence,

$$z(t) = \frac{1}{t}\left(b \pm \frac{t}{2}\sqrt{a^2 + 4ct^2} \pm \frac{a^2}{4\sqrt{c}}\ln|2\sqrt{c}t + \sqrt{a^2 + 4ct^2}|\right)$$

and the general solution of (24) is

$$\varphi(t) = \pm\frac{1}{t}\sqrt{\left(b \pm \frac{t}{2}\sqrt{a^2 + 4ct^2} \pm \frac{a^2}{4\sqrt{c}}\ln|2\sqrt{c}t + \sqrt{a^2 + 4ct^2}|\right)^2 - t^2}.$$

If $c < 0$, i.e. the mean curvature vector field H is timelike, then $\int \sqrt{a^2 + 4ct^2}\,dt = \frac{t}{2}\sqrt{a^2 - 4ct^2} + \frac{a^2}{4\sqrt{-c}}\arcsin\frac{2\sqrt{-c}t}{a}$. Hence, the function $z(t)$ is given by

$$z(t) = \frac{1}{t}\left(b \pm \frac{t}{2}\sqrt{a^2 - 4ct^2} \pm \frac{a^2}{4\sqrt{-c}}\arcsin\frac{2\sqrt{-c}t}{a}\right)$$

and the general solution of (24) is

$$\varphi(t) = \pm\frac{1}{t}\sqrt{\left(b \pm \frac{t}{2}\sqrt{a^2 - 4ct^2} \pm \frac{a^2}{4\sqrt{-c}}\arcsin\frac{2\sqrt{-c}t}{a}\right)^2 - t^2}.$$

$\square$

Now, let $\mathcal{M}'_b$ be a meridian surface on $\mathcal{M}^I$ defined by (7). Hence, the mean curvature vector field H of $\mathcal{M}'_b$ is given by formula (9) and

$$\langle H, H \rangle = \frac{\kappa^2(f'^2 - 1) - \left(ff'' + (f')^2 - 1\right)^2}{4f^2(f'^2 - 1)}.$$

The classification of the meridian surfaces of type $\mathcal{M}'_b$ for which $\langle H, H \rangle = c = const, \ c \neq 0$ is given in the following theorem:

Theorem 11. *Let $\mathcal{M}'_b$ be a meridian surface on $\mathcal{M}^I$ defined by (7). Then $\mathcal{M}'_b$ has non-zero constant mean curvature, i.e. $\langle H, H \rangle = c = const, \ c \neq 0$, if and*

only if the curve c has constant curvature $\kappa = a = const, \; a \neq 0$ *and the meridian curve m is determined by* $f' = \varphi(f)$, *where*

$$\varphi(t) = \pm\frac{1}{t}\sqrt{\left(b \pm \frac{t}{2}\sqrt{a^2 - 4ct^2} \pm \frac{a^2}{4\sqrt{c}}\arcsin\frac{2\sqrt{c}\,t}{a}\right)^2 + t^2}, \quad \text{if } c > 0,$$

$$\varphi(t) = \pm\frac{1}{t}\sqrt{\left(b \pm \frac{t}{2}\sqrt{a^2 - 4ct^2} \pm \frac{a^2}{4\sqrt{-c}}\ln|2\sqrt{-c}t + \sqrt{a^2 - 4ct^2}|\right)^2 + t^2}, \quad \text{if } c < 0,$$

$b = const,$ *and* $g(u)$ *is defined by* $g' = \sqrt{f'^2 - 1}$.

The proof of this theorem is similar to the proof of Theorem 10.

In the next theorem we give the classification of the meridian surfaces of type $\mathcal{M}''$ for which $\langle H, H\rangle = c = const, \; c \neq 0$.

Theorem 12. *Let* $\mathcal{M}''$ *be a meridian surface on* $\mathcal{M}^{II}$ *defined by* (10). *Then* $\mathcal{M}''$ *has non-zero constant mean curvature, i.e.* $\langle H, H\rangle = c = const, \; c \neq 0,$ *if and only if the curve c has constant curvature* $\kappa = a = const, \; a \neq 0$ *and the meridian curve m is determined by* $f' = \varphi(f)$, *where*

$$\varphi(t) = \pm\frac{1}{t}\sqrt{t^2 - \left(b \mp \frac{t}{2}\sqrt{a^2 - 4ct^2} \mp \frac{a^2}{4\sqrt{c}}\arcsin\frac{2\sqrt{c}\,t}{a}\right)^2}, \quad \text{if } c > 0,$$

$$\varphi(t) = \pm\frac{1}{t}\sqrt{t^2 - \left(b \mp \frac{t}{2}\sqrt{a^2 - 4ct^2} \mp \frac{a^2}{4\sqrt{-c}}\ln|2\sqrt{-c}t + \sqrt{a^2 - 4ct^2}|\right)^2}, \quad \text{if } c < 0,$$

$b = const,$ *and* $g(u)$ *is defined by* $g' = \sqrt{1 - f'^2}$.

At the end of this section we give the classification of the meridian surfaces of type $\mathcal{M}'''$ with non-zero constant mean curvature.

Theorem 13. *Let* $\mathcal{M}'''$ *be a meridian surface on* $\mathcal{M}^{III}$ *defined by* (14). *Then* $\mathcal{M}'''$ *has non-zero constant mean curvature* $\langle H, H\rangle = \varepsilon c, \; c = const, \; c \neq 0,$ *if and only if the curve c has constant curvature* $\kappa = a = const, \; a \neq 0$ *and the meridian curve m is determined by* $f' = \varphi(f)$ *where*

$$\varphi(t) = \frac{1}{t}\left(b \pm \frac{t}{2}\sqrt{a^2 + 4ct^2} \pm \frac{a^2}{4\sqrt{c}}\ln|2\sqrt{c}t + \sqrt{a^2 + 4ct^2}|\right), \quad \text{if } c > 0,$$

$$\varphi(t) = \frac{1}{t}\left(b \pm \frac{t}{2}\sqrt{a^2 - 4ct^2} \pm \frac{a^2}{4\sqrt{-c}}\arcsin\frac{2\sqrt{-c}\,t}{a}\right), \quad \text{if } c < 0,$$

$b = const,$ *and* $g(u)$ *is defined by* $g' = \frac{1}{2\varepsilon f'}$.

Proof. The mean curvature vector field of the meridian surface $\mathcal{M}'''$ is given by (19). So, the condition on $\mathcal{M}'''$ to have non-zero constant mean curvature $\langle H, H\rangle = \varepsilon c, \; c = const \neq 0$ is equivalent to the equality

$$\frac{\left(ff'' + f'^2\right)^2 - \kappa^2 f'^2}{4f^2 f'^2} = c.$$

The last equality can be written as

$$\frac{\left(ff'' + f'^2\right)^2 - 4cf^2 f'^2}{f'^2} = \kappa^2. \tag{25}$$

Since the left-hand side of (25) is a function of u, the right-hand side of (25) is a function of v, we obtain

$$\kappa = a, \quad a = const \neq 0;$$
$$\frac{\left(ff'' + f'^2\right)^2 - 4cf^2 f'^2}{f'^2} = a^2.$$

Hence, the meridian m is determined by the solutions of the following differential equation:

$$ff'' + f'^2 = \pm f'\sqrt{a^2 + 4cf^2}.$$

Setting $f' = \varphi(f)$ we obtain that the function $\varphi = \varphi(t)$ is a solution of the equation:

$$\varphi' + \frac{1}{t}\varphi = \pm\frac{\sqrt{a^2 + 4ct^2}}{t}. \tag{26}$$

If $c > 0$, then the general solution of (26) is

$$\varphi(t) = \frac{1}{t}\left(b \pm \frac{t}{2}\sqrt{a^2 + 4ct^2} \pm \frac{a^2}{4\sqrt{c}}\ln|2\sqrt{c}t + \sqrt{a^2 + 4ct^2}|\right).$$

If $c < 0$, then the general solution of (26) is given by

$$\varphi(t) = \frac{1}{t}\left(b \pm \frac{t}{2}\sqrt{a^2 - 4ct^2} \pm \frac{a^2}{4\sqrt{-c}}\arcsin\frac{2\sqrt{-c}\,t}{a}\right).$$

$\square$

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