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Research article

Chebyshev type inequalities involving extended generalized fractional integral operators

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Abstract: In this paper, mainly by using the extended generalized fractional integral operator that involve a further extension of Mittag-Leffler function in the kernel, we obtain several fractional Chebyshev type integral inequalities. So, results of Dahmani *et al.* from [4] are generalized. Also, it is point out that new results are obtained for different fractional integral operators with the help of special selection of parameters.

Keywords: Chebyshev inequality; fractional integral operators
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1. Introduction

In 1882, Chebyshev proved on interesting and useful integral inequality as follows:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.1}$$

where f and g are two integrable and synchronous functions on $[a,b]$. Here two functions f and g are called synchronous on $[a,b]$, if

$$(f(x)-f(y))(g(x)-g(y)) \geq 0 \quad (x,y \in [a,b]).$$

The inequality (1.1) that is well known as Chebyshev inequality has many applications in diverse research subjects such as numerical quadrature, transform theory, probability, existence of solutions of differential equations and statistical problems. Therefore, many researchers have given considerable attention to this inequality, (see [3, 10–12, 18–22].

On the other hand, one of the methods used to generalize inequalities is fractional calculus. In this context, firstly, in 2009, Chebyshev inequality involving Riemann-Liouville fractional integrals is presented as the following:

Theorem 1.1. *([2]) Let f and g be two synchronous function on* $[0,\infty)$*. Then for all* $t>0,\ \ \alpha>0$*, we have:*

$$J^{\alpha}(f,g)(t)\geq\frac{\Gamma(\alpha+1)}{t^{\alpha}}J^{\alpha}f(t)J^{\alpha}g(t) \tag{1.2}$$

where $J^{\alpha}f(t)$ *denotes Riemann-Liouville fractional integral operator of a function* $f(t)$ *and* Γ *is the Gamma function such that these are defined as follows (see e.g. [17]).*

Let $f\in L[a,b]$*. The Riemann-Liouville fractional integrals* $J^{\alpha}_{a^+}f$ *and* $J^{\alpha}_{b^-}f$ *of order* $\alpha>0$ *are defined by*

$$\begin{aligned} J^{\alpha}_{a^+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x>a \\ J^{\alpha}_{b^-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x<b \end{aligned}$$

respectively. Also the gamma function is defined by

$$\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt.$$

Here is $J^0_{a^+}f(x)=J^0_L-f(x)=f(x)$*. In the case of* $\alpha=1$*, the this fractional integral reduces to the classical integral.*

After this study of Belarbi and Dahmani, generalizations of Chebyshev inequality were obtained for the different types of fractional integral operator with similar technique. (see, e.g. [2,5–7,14,23–25,28].

Recently, the new generalizations of the Riemann-Liouville fractional integral operator, have been described with the help of various extensions of the Mittag-Leffler function. Now lets give some of these operators which we will need in the second section.

Definition 1.1. *([13]) Let* $\alpha,\beta,\rho,\lambda\in\mathbb{C}$*,* $Re(\alpha)>0$ *and* $Re(\beta)>0$*. Let* $f\in L[a,b]$ *and* $x\in[a,b]$*. Then the fractional integral operator* $\epsilon(\alpha,\beta,\rho,\lambda)$ *defined by Prabhakar is as the following:*

$$\epsilon(\alpha,\beta,\rho,\lambda)f(x)=\int_a^x (x-t)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha}f(t)dt$$

where

$$E^{\rho}_{\alpha,\beta}=\sum_{n=0}^{\infty}\frac{(\rho)_n z^n}{\Gamma(\alpha n+\beta)n!}$$

and Γ *is the Gamma function.*

Definition 1.2. *([26]) Let* $z,\beta,\gamma,\omega\in\mathbb{C}$*,* $Re(\alpha)>max\{0,Re(\kappa)-1\}$*,* $min\{Re(\beta),Re(\kappa)\}>0$*. Let* $f\in L[a,b]$ *and* $x\in[a,b]$*. Then the fractional integral operator* $\epsilon^{\omega;\gamma,\kappa}_{a^+;\alpha,\beta}\varphi$ *defined by Srivastava and Tomowski is as the following:*

$$(\epsilon^{\omega;\gamma,\kappa}_{a^+;\alpha,\beta}\varphi)(x)=\int_a^x (x-t)^{\beta-1}E^{\gamma,\kappa}_{\alpha,\beta}[\omega(x-t)^{\alpha}]\varphi(t)dt \quad (x>a)$$

where

$$E^{\gamma,\kappa}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}$$

and Γ *is the Gamma function.*

Definition 1.3. *([16]) Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$*,* $\min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)\} > 0$*,* $p, q > 0$ *and* $q \leq Re(\alpha) + p$*. Let* $f \in L[a, b]$ *and* $x \in [a, b]$*. Then the fractional integral operator* $\epsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}$ *defined by Salim and Faraj is as the following:*

$$\epsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}\varphi(x) = \int_a^x (x - t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x - t)^{\alpha})\varphi(t)dt$$

where

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{\gamma_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}$$

and Γ *is the Gamma function.*

Definition 1.4. *([15]) Let* $p \geq 0$*,* $q > 0$*,* $\omega, \delta, \lambda, \sigma, c, \rho \in \mathbb{C}$*,* $Re(c) > 0$*,* $Re(\rho) > 0$ *and* $Re(\sigma) > 0$*. Let* $f \in L[a, b]$ *and* $x \in [a, b]$*. Then the fractional integral operator* $(\epsilon^{\omega,\delta,q,c}_{a^+,\rho,\sigma} f)$ *defined by Rahman et al. is as the following:*

$$(\epsilon^{\omega,\delta,q,c}_{a^+,\rho,\sigma} f)(x) = \int_a^x (x - \tau)^{\sigma-1} E^{\delta,q,c}_{p,\sigma}(\omega(x - \tau)^{\rho}; p) f(\tau)d\tau$$

where

$$E^{\delta,q,c}_{\rho,\sigma}(z; p) = \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{n!}$$

and $B_p(x, y)$ *is an extension of Beta function defined in [15]*

$$B_p(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} e^{-\frac{p}{t(1-t)}} dt \quad x, y, p > 0 \tag{1.3}$$

where $Re(p) > 0$*,* $Re(x) > 0$ *and* $Re(y) > 0$*. Also, here B is familiar Beta function as follows:*

$$B\,(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} = \int_0^1 t^{a-1}\,(1 - t)^{b-1}\,dt, \quad a, b > 0. \tag{1.4}$$

Definition 1.5. *([1]) Let* $\omega, \alpha, \beta, \sigma, \delta, c \in \mathbb{C}$*,* $Re(\alpha), Re(\beta), Re(\sigma), Re(\delta), Re(c) > 0$ *with* $p \geq 0$*,* $q > 0$ *and* $0 < r \leq q + Re(\alpha)$*. Let* $f \in L_1[a, b]$ *and* $x \in [a, b]$*. Then the generalized fractional integral operator* $\epsilon^{\omega,\delta,q,r,c}_{a^+,\alpha,\beta,\sigma} f$ *is defined by*

$$\left(\epsilon^{\omega,\delta,q,r,c}_{a^+,\alpha,\beta,\sigma} f\right)(x; p) = \int_a^x (x - t)^{\beta-1} E^{\delta,q,r,c}_{\alpha,\beta,\sigma}(\omega(x - t)^{\alpha}; p) f(t)dt \tag{1.5}$$

where

$$E^{\delta,q,r,c}_{\alpha,\beta,\sigma}(z; p) = \sum_{n=0}^{\infty} \frac{B_p(\delta + nq, c - \delta)}{B(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\sigma)_{nr}}$$

and B_p *and* B *is as* (1.3) *and* (1.4) *respectively. For further information about this operator, (see [1, 8, 9, 27]).*

2. Main results

Theorem 2.1. *Let t be a positive function on $[0,\infty]$ and let f and g be two differentiable functions on $[0,\infty]$. If $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty])$, $r > 1$, $r^{-1} + s^{-1} = 1$, then for all $x > 0$, $\alpha > 0$, $\beta > 0$, we have*

$$\begin{aligned}
& 2\left|(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tfg)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} t)(x;p) - (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tf)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tg)(x;p)\right| \\
\leq\ & \|f'\|_r \|g'\|_s \int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho| t(\tau)t(\rho) \\
\times\ & E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big) E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big) d\tau d\rho \\
\leq\ & \|f'\|_r \|g'\|_s x(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} t(x;p))^2.
\end{aligned} \tag{2.1}$$

Proof. Let f and g be two functions satisfying the conditions of Theorem 2.1 and let t be a positive function on $[0,\infty]$, Define

$$H(\tau,\rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)); \tau,\rho \in (0,x), x > 0. \tag{2.2}$$

Multiplying (2.2) by

$$(x-\tau)^{(\beta-1)} E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)\ t(\tau); \tau \in (0,x)$$

and integrating the resulting identity with respect to τ from 0 x, we can state that

$$\begin{aligned}
& \int_0^x (x-\tau)^{(\beta-1)} E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big) t(\tau) H(\tau,\rho) d\tau \\
=\ & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tfg)(x;p) - f(\rho)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tg)(x;p) \\
-\ & g(\rho)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tf)(x;p) + f(\rho)g(\rho)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} t)(x;p)
\end{aligned} \tag{2.3}$$

Now, multiplying (2.3) by

$$(x-\rho)^{(\beta-1)} E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big) t(\rho); \rho \in (0,x)$$

and integrating the resulting identity with respect to ρ over $(0,x)$, we can write

$$\begin{aligned}
& \int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)} E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big) \\
\times\ & E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big) t(\tau)t(\rho) H(\tau,\rho) d\tau d\rho \\
=\ & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tfg)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} t)(x;p) - (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tf)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tg)(x;p) \\
-\ & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tg)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tf)(x;p) + (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} tfg)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma} t)(x;p)
\end{aligned}$$

Consequently,

$$\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)} E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)$$

$$\times \quad E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)H(\tau,\rho)d\tau d\rho$$
$$= \quad 2\Big((\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tfg)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(x;p)-(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tf)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tg)(x;p)\Big) \tag{2.4}$$

On the other and, we have

$$H(\tau,\rho) := \int_\tau^\rho \int_\tau^\rho f'(y)g'(z)dydz. \tag{2.5}$$

Using Hölder inequality for double integral, we can write

$$|H(\tau,\rho)| \le \Big|\int_\tau^\rho \int_\tau^\rho |f'(y)|^r dydz\Big|^{r^{-1}} \Big|\int_\tau^\rho \int_\tau^\rho |g'(z)|^s dydz\Big|^{s^{-1}} \tag{2.6}$$

Since,

$$\Big|\int_\tau^\rho \int_\tau^\rho |f'(y)|^r dydz\Big|^{r^{-1}} = |\tau-\rho|^{r^{-1}}\Big|\int_\tau^\rho |f'(y)|^r dy\Big|^{r^{-1}} \tag{2.7}$$

and

$$\Big|\int_\tau^\rho \int_\tau^\rho |g'(z)|^s dydz\Big|^{s^{-1}} = |\tau-\rho|^{s^{-1}}\Big|\int_\tau^\rho |g'(z)|^s dz\Big|^{s^{-1}} \tag{2.8}$$

then, we can estimate H as follows:

$$|H(\tau,\rho)| \le |\tau-\rho|\Big|\int_\tau^\rho |f'(y)|^r dy\Big|^{r^{-1}}\Big|\int_\tau^\rho |g'(z)|^s dz\Big|^{s^{-1}} \tag{2.9}$$

On the other hand, we have

$$\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)$$
$$\times \quad E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big|d\tau d\rho$$
$$\le \quad \int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho|t(\tau)t(\rho)E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)$$
$$\times \quad E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)$$
$$\times \quad \Big|\int_\tau^\rho |f'(y)|^r dy\Big|^{r^{-1}}\Big|\int_\tau^\rho |g'(z)|^s dz\Big|^{s^{-1}} d\tau d\rho \tag{2.10}$$

Applying again Hölder inequality to the right-hand side of (2.10), we can state that

$$\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)$$
$$\times \quad E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big|d\tau d\rho$$
$$\le \quad \Big[\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho|t(\tau)t(\rho)E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)$$
$$\times \quad E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)\Big|\int_\tau^\rho |f'(y)|^r dy\Big|d\tau d\rho\Big]^{r^{-1}}$$

$$\times \quad \Big[\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho| t(\tau)t(\rho) E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)$$

$$\times \quad E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)\Big|\int_\tau^\rho |g'(z)|^s dz\Big| d\tau d\rho\Big]^{s^{-1}}.$$

Now, using the fact the

$$\Big|\int_\tau^\rho |f'(y)|^r dy\Big| \le \|f'\|_r^r, \quad \Big|\int_\tau^\rho |g'(z)|^s dz\Big| \le \|g'\|_s^s \tag{2.11}$$

we obtain

$$\begin{aligned}
&\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)} E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big| d\tau d\rho\\
\le\ & \Big[\|f'\|_r^r \int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho| t(\tau)t(\rho)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)d\tau d\rho\Big]^{r^{-1}}\\
\times\ & \Big[\|g'\|_s^s \int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho| t(\tau)t(\rho)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)d\tau d\rho\Big]^{s^{-1}}.
\end{aligned} \tag{2.12}$$

From (2.12), we get

$$\begin{aligned}
&\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)} E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big| d\tau d\rho\\
\le\ & \|f'\|_r\|g'\|_s\Big[\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho| t(\tau)t(\rho)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)d\tau d\rho\Big]^{r^{-1}}\\
\times\ & \Big[\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho| t(\tau)t(\rho)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)d\tau d\rho\Big]^{s^{-1}}.
\end{aligned} \tag{2.13}$$

Since $r^{-1} + s^{-1} = 1$, then we have

$$\begin{aligned}
&\int_0^x \int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)} E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big| d\tau d\rho
\end{aligned} \tag{2.14}$$

$$\leq \|f'\|_r\|g'\|_s\Big[\int_0^x\int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}|\tau-\rho|t(\tau)t(\rho)$$
$$\times E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)d\tau d\rho\Big].$$

By the relations (2.4) and (2.14) and using the properties of the modulus, we get the first inequality in (2.1). We have

$$0\leq\tau\leq x,\quad 0\leq\rho\leq x.$$

Hence,

$$0\leq|\tau-\rho|\leq x.$$

Therefore, we have

$$\int_0^x\int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)$$
$$\times E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big|d\tau d\rho$$
$$\leq \|f'\|_r\|g'\|_s x\Big[\int_0^x\int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\beta-1)}$$
$$\times E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\rho)^{\alpha};p\Big)t(\tau)t(\rho)d\tau d\rho\Big]$$
$$= \|f'\|_r\|g'\|_s\; x\big(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t(x)\big)^2. \tag{2.15}$$

Theorem (2.1) is thus proved.

□

In Theorem 2.1, if we set $t(x) = 1$, we arrive at the following corollary :

Corollary 2.1. *Let f and g be two differentiable functions on $[0,\infty]$. If $f'\in L_r([0,\infty])$, $g'\in L_s([0,\infty])$, $r>1$, $r^{-1}+s^{-1}=1$, then for all $x>0$, $\alpha>0$, $\beta>0$, we have:*

$$\Big|(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}fg)(x;p)-\frac{1}{(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma})(1)}(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}f)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}g)(x;p)\Big|$$
$$\leq \frac{1}{2}(\|f'\|_r\|g'\|_s x(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma})(1)). \tag{2.16}$$

Corollary 2.2. *For different choices of parameters in (2.1) we can establish the corresponding fractional integral inequalities such as*

(i) setting $p = 0$, we get Chebyshev inequality for the Salim-Faraj fractional integral operator, defined in [16],

(ii) setting $\sigma = r = 1$, we get Chebyshev inequality for the fractional integral operator defined by Rahman et al. in [15],

(iii) setting $p = 0$ and $\sigma = r = 1$, we get Chebyshev inequality for the Srivastava-Tomovski fractional integral operator defined in [26],

(iv) setting $p = 0$ and $\sigma = r = q = 1$, we get Chebyshev inequality for the Prabhakar fractional integral operator defined in [13].

Remark 2.1. *In (2.1) setting $p=\omega=0$, we get the inequality (3.1) in [4].*

Theorem 2.2. *Let t be a positive function on $[0,\infty]$ and let f and g be two differentiable functions on $[0,\infty]$. If $f'\in L_r([0,\infty])$, $g'\in L_s([0,\infty])$, $r>1$, $r^{-1}+s^{-1}=1$, then for all $x>0$, $\alpha>0$, $\beta>0$, $\lambda>0$, $\theta>0$, we have*

$$\begin{aligned}
& \Big|(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tfg)(x;p)+(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tfg)(x;p)\\
- & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tf)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tg)(x;p)-(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tf)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tg)(x;p)\Big|\\
\leq & \|f'\|_r\|g'\|_s\int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}|\tau-\rho|t(\tau)t(\rho)\\
\times & E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)E^{\delta,q,r,c}_{\lambda,\theta,p}\Big(\omega(x-\rho)^{\lambda};p\Big)d\tau d\rho\\
\leq & \|f'\|_r\|g'\|_s x(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t)(x;p). \qquad (2.17)
\end{aligned}$$

Proof. Using the identity (2.3), we can state that

$$\begin{aligned}
& \int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times & E^{\delta,q,r,c}_{\lambda,\theta,p}\Big(\omega(x-\rho)^{\lambda};p\Big)t(\tau)t(\rho)H(\tau,\rho)d\tau d\rho\\
= & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tfg)(x;p)+(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tfg)(x;p)\\
- & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tf)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tg)(x;p)-(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}tf)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}tg)(x;p). \qquad (2.18)
\end{aligned}$$

From the relation (2.9), we can obtain the following estimation

$$\begin{aligned}
& \int_0^x(x-\tau)^{(\beta-1)}E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)t(\tau)\Big|H(\tau,\rho)\Big|d\tau\\
\leq & \int_0^x(x-\tau)^{(\beta-1)}|\tau-\rho|E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)t(\tau)\\
\times & \Big|\int_\tau^\rho|f'(y)|^r dy\Big|^{r^{-1}}\Big|\int_\tau^\rho|g'(z)|^s dz\Big|^{s^{-1}}d\tau \qquad (2.19)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times & E^{\delta,q,r,c}_{\lambda,\theta,p}\Big(\omega(x-\rho)^{\lambda};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big|d\tau d\rho\\
\leq & \int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}|\tau-\rho|E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times & E^{\delta,q,r,c}_{\lambda,\theta,p}\Big(\omega(x-\rho)^{\lambda};p\Big)t(\tau)t(\rho)\Big|\int_\tau^\rho|f'(y)|^r dy\Big|^{r^{-1}}\Big|\int_\tau^\rho|g'(z)|^s dz\Big|^{s^{-1}}d\tau d\rho. \qquad (2.20)
\end{aligned}$$

Applying Hölder inequality for double integral to the right-hand side of (2.20), yields

$$\int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}E^{\delta,q,r,c}_{\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)$$

$$
\begin{aligned}
\times\quad & E_{\lambda,\theta,p}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\lambda};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big|d\tau d\rho\\
\leq\quad & \Big[\int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}|\tau-\rho|E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times\quad & E_{\lambda,\theta,p}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\lambda};p\Big)t(\tau)t(\rho)\Big|\int_\tau^\rho|f'(y)|^r dy\Big|d\tau d\rho\Big]^{r^{-1}}\\
\times\quad & \Big[\int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}|\tau-\rho|E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times\quad & E_{\lambda,\theta,p}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\lambda};p\Big)t(\tau)t(\rho)\Big|\int_\tau^\rho|g'(z)|^s dz\Big|d\tau d\rho\Big]^{s^{-1}}. \qquad (2.21)
\end{aligned}
$$

By (2.11) and (2.21), we get

$$
\begin{aligned}
& \int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)\\
\times\ & E_{\lambda,\theta,p}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\lambda};p\Big)t(\tau)t(\rho)\Big|H(\tau,\rho)\Big|d\tau d\rho\\
\leq\ & \|f'\|_r\|g'\|_s\int_0^x\int_0^x(x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}|\tau-\rho|t(\tau)t(\rho)\\
\times\ & E_{\alpha,\beta,\sigma}^{\delta,q,r,c}\Big(\omega(x-\tau)^{\alpha};p\Big)E_{\lambda,\theta,p}^{\delta,q,r,c}\Big(\omega(x-\rho)^{\lambda};p\Big)d\tau d\rho. \qquad (2.22)
\end{aligned}
$$

Using (2.18) and (2.22) and the properties of modulus, we get the first inequality in (2.17). □

Corollary 2.3. *For different choices of parameters in (2.17) we can establish the corresponding fractional integral inequalities such as*

(i) setting $p = 0$*, we get Chebyshev inequality for the Salim-Faraj fractional integral operator, defined in [16],*

(ii) setting $\sigma = r = 1$*, we get Chebyshev inequality for the fractional integral operator defined by Rahman et al. in [15],*

(iii) setting $p = 0$ *and* $\sigma = r = 1$*, we get Chebyshev inequality for the Srivastava-Tomovski fractional integral operator defined in [26],*

(iv) setting $p = 0$ *and* $\sigma = r = q = 1$*, we get Chebyshev inequality for the Prabhakar fractional integral operator defined in [13].*

Remark 2.2. *Applying Theorem 2.2 for* $\beta = \theta$*,* $\alpha = \lambda$*, we obtain theorem 2.1.*

Remark 2.3. *In (2.17) setting* $p = \omega = 0$*, we get the inequality (3.17) in [4].*

Conflict of interest

The authors declare there is no conflict of interest in this paper.

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