A STUDY OF THREE-DIMENSIONAL PARACONTACT $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-SPACES

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Abstract. This paper is a study of three-dimensional paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifolds. Three dimensional paracontact metric manifolds whose Reeb vector field ξ is harmonic are characterized. We focus on some curvature properties by considering the class of paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifolds under a condition which is given at Definition 3.1. We study properties of such manifolds according to the cases $\tilde{\kappa}>-1$, $\tilde{\kappa}=-1$, $\tilde{\kappa}<-1$ and construct new examples of such manifolds for each case. We also show that the existence of paracontact metric $(-1,\tilde{\mu}\neq 0,\tilde{\nu}\neq 0)$ spaces with dimension greater than 3 such that $\tilde{h}^2=0$ but $\tilde{h}\neq 0$.

1. Introduction

Paracontact manifolds are smooth manifolds of dimension $2n+1$ endowed with a 1-form η, a vector field ξ and a field of endomorphisms of tangent spaces $\tilde{\varphi}$ such that $\eta(\xi)=1$, $\tilde{\varphi}^2=I-\eta\otimes\xi$ and $\tilde{\varphi}$ induces an almost paracomplex structure on the codimension 1 distribution defined by the kernel of η (see § 2 for more details). In addition, if the manifold is endowed with a pseudo-Riemannian metric $\tilde{g}$ of signature $(n+1,n)$ satisfying

$$\tilde{g}(\tilde{\varphi}X,\tilde{\varphi}Y)=-\tilde{g}(X,Y)+\eta(X)\eta(Y),\quad d\eta(X,Y)=\tilde{g}(X,\tilde{\varphi}Y),$$

(M,η) becomes a contact manifold and $(\tilde{\varphi},\xi,\eta,\tilde{g})$ is said to be a paracontact metric structure on M. The study of paracontact geometry was started by Kaneyuki and Williams in [18] and then it was continued by many other authors. A systematic study of paracontact metric manifolds, and some subclasses like para-Sasakian manifolds, was carried out in a striking paper of Zamkovoy [31]. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics (cf. e.g. [1], [2], [12], [14], [15]).

In [13], Closset, Dumitrescu, Festuccia, and Komargodski constructed supersymmetric field theories on Riemannian three manifolds, considering three dimensional theories with $\mathcal{N}=2$ supersymmetry. They proved that the supersymmetric field theory on three manifold M possesses a single supercharge if and only if M admits an almost contact metric structure that satisfies certain integrability conditions. Recently, Willett [29] studied the localization of three dimensional $\mathcal{N}=2$ supersymmetric theories on compact manifolds.

As known every orientable Riemannian three-manifold admits a metric-compatible almost contact structure and three dimensional unit sphere S^3 has a Sasakian structure. In [22] Markellos and Tsichlias constructed new (κ,μ) contact metric structures (non-Sasakian) on the unit sphere S^3.

On the other hand, in [11] it was proved (cf. Theorem 2.1 below) that any (non-Sasakian) contact (κ,μ)-space carries a canonical paracontact metric structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$ whose Levi-Civita connection satisfies a condition formally similar to contact case

$$\tilde{R}(X,Y)\xi=\tilde{\kappa}\left(\eta\left(Y\right)X-\eta\left(X\right)Y\right)+\tilde{\mu}(\eta\left(Y\right)\tilde{h}X-\eta\left(X\right)\tilde{h}Y),\tag{1.1}$$

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where $2\tilde{h} := \mathcal{L}_\xi \tilde{\varphi}$ and, in this case, $\tilde{\kappa} = (1-\mu/2)^2 + \kappa - 2$, $\tilde{\mu} = 2$. By [11] and [22], S^3 will have paracontact metric structure.

A $(2n+1)$-dimensional paracontact metric manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ whose curvature tensor satisfies (1.1), is called *paracontact* $(\tilde{\kappa}, \tilde{\mu})$*-manifold*. The class of paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds is very large. It contains para-Sasakian manifolds, as well as those paracontact metric manifolds satisfying $\tilde{R}(X,Y)\xi = 0$ for all $X, Y \in \Gamma(TM)$ (recently studied in [32]). But, unlike in the contact Riemannian case, a paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold such that $\tilde{\kappa} = -1$ in general is not para-Sasakian. In fact, there are paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds such that $\tilde{h}^2 = 0$ (which is equivalent to take $\tilde{\kappa} = -1$) but with $\tilde{h} \neq 0$. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric (κ, μ)-spaces the constant κ can not be greater than 1, here we have no restriction for the constants $\tilde{\kappa}$ and $\tilde{\mu}$. It should be also remarked that contact metric (κ, μ, ν)-spaces of dimension greater than 3 is either a Sasakian manifold or a (κ, μ)-contact metric manifold. But the authors provided an example of paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\upsilon})$-manifold such that $\tilde{\kappa} = -1$, (and $\tilde{h}^2 = 0$) but with $\tilde{\mu} \neq 0, \tilde{\nu} \neq 0$, $\tilde{h} \neq 0$, and $n > 1$.

Cappelletti Montano et al.[12] showed in that there is a kind of duality between those manifolds and contact metric (κ, μ)-spaces and also proved that ξ is a Ricci eigenvector of paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds.

In [8], G.Calvaruso and D. Perrone proved that ξ is *harmonic* if and only if ξ is an eigenvector of the Ricci operator for contact semi-Riemannian manifolds.

It turns out that there exists a motivation to study harmonic maps in contact semi-Riemannian and paracontact geometry.

Let (M, g) be smooth, oriented, connected pseudo-Riemannian manifold and (TM, g^S) its tangent bundle endowed with the Sasaki metric (also referred to as Kaluza-Klein metric in Mathematical Physics) g^S. By definition, the *energy* of a smooth vector field V on M is the energy corresponding $V : (M, g) \to (TM, g^s)$. When M is compact, the *energy* of V is determined by

$$E(V) = \frac{1}{2}\int_M (tr_g V^* g^s) dv = \frac{n}{2} vol(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 dv.$$

The non-compact case, one can take into account over relatively compact domains. It can be shown that $V : (M, g) \to (TM, g^s)$ is harmonic map if and only if

$$tr\,[R(\nabla.V, V).] = 0,\ \nabla^*\nabla V = 0, \tag{1.2}$$

where

$$\nabla^*\nabla V = \sum_i \varepsilon_i (\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) \tag{1.3}$$

is the rough Laplacian with respect to a pseudo-orthonormal local frame $\{e_1, ..., e_n\}$ on (M, g) with $g(e_i, e_i) = \varepsilon_i = \pm 1$ for all indices $i = 1, ..., n$.

If (M, g) is a compact Riemannian manifold, only parallel vector fields define harmonic maps.

Next, for any real constant $\rho \neq 0$, let $\chi^\rho(M) = \left\{ W \in \chi(M) : \|W\|^2 = \rho \right\}$.We consider vector fields $V \in \chi^\rho(M)$ which are critical points for the energy functional $E\mid_{\chi^\rho(M)}$, restricted to vector fields of the same length. The Euler-Lagrange equations of this variational condition yield that V is a harmonic vector field if and only if

$$\nabla^*\nabla V \text{ is collinear to } V. \tag{1.4}$$

This characterization is well known in the Riemannian case ([3, 23, 25]). Using same arguments in pseudo-Riemannian case, G. Calvaruso [5] proved that same result is still valid for vector fields of constant length, if it is not lightlike.

Let $T_1 M$ denote the unit tangent sphere bundle over M, and again by g^S the metric induced on $T_1 M$ by the Sasaki metric of TM. Then, it is shown that in [3], the map on $V : (M, g) \to (T_1 M, g^s)$ is harmonic if V is a harmonic vector field and the additonial condition

$$tr[R(\nabla .V, V).] = 0 \tag{1.5}$$

is satisfied. G. Calvaruso [5] also investigated harmonicity properties for left-invariant vector fields on three-dimensional Lorentzian Lie groups, obtaining several classification results and new examples of critical points of energy functionals. In [7], he studied harmonicity properties of vector fields on four-dimensional pseudo-Riemannian generalized symmetric spaces. Moreover, he gave a complete classification of three-dimensional homogeneous paracontact metric manifolds in [6]. Recently, G.Calvaruso and D.Perrone [9] proved that all three-dimensional homogeneous paracontact metric manifolds are *H-paracontact*, that is, paracontact metric manifolds whose characteristic vector field ξ is harmonic.

In this paper, we study harmonicity of the characteristic vector field of three-dimensional paracontact metric manifolds and give a characterization of $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifolds.

Overview: Here is the plan of the paper: § 2 is devoted to preliminaries. In § 3, we study the common properties of paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifolds (see § 3 for definition) for the cases $\tilde{\kappa} < -1$, $\tilde{\kappa} = -1$, $\tilde{\kappa} > -1$. Beside the other results, we prove for instance that while the values of $\tilde{\kappa}$, $\tilde{\mu}$ and $\tilde{\nu}$ change, the paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\upsilon})$-manifolds remain unchanged under $\mathcal{D}$-homothetic deformations. Moreover we prove that in dim $M > 3$, paracontact metric $(\tilde{\kappa} \neq -1, \tilde{\mu}, \tilde{\nu})$-manifolds must be paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds.

We reserve § 4, for the following two main theorems of the paper:

Theorem 1.1. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact metric manifold. ξ is a harmonic vector field if and only if the characteristic vector field ξ is an eigenvector of the Ricci operator.*

Theorem 1.2. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact metric manifold. If the characteristic vector field ξ is harmonic vector field then the paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold always exists on every open and dense subset of M. Conversely, if M is a paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold then the characteristic vector field ξ is harmonic vector field.*

We also show that a paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold with $\tilde{\kappa} = -1$ is not necessary para-Sasakian (see the tensor $\tilde{h}$ has the canonical form (II)). As stated above, this case shows important difference with the contact Riemannian case. So we can construct non-trival examples of non-para-Sasakian manifold with $\tilde{\kappa} = -1$. Also one could find examples about paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifolds according to the cases $\tilde{\kappa} > -1$, $\tilde{\kappa} < -1$.

In § 5, we give a relation between non-Sasakian $(\kappa, \mu, \nu = const.)$-contact metric manifold with the Boeckx invariant $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$ is constant along the integral curves of ξ i.e. $\xi(I_M) = 0$ and 3-dimensional paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold.

2. Preliminaries

An $(2n+1)$-dimensional smooth manifold M is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field $\tilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1$, $\tilde{\varphi}^2 = I - \eta \otimes \xi$,

(ii) the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e. the ± 1-eigendistributions, $\mathcal{D}^{\pm} := \mathcal{D}_{\tilde{\varphi}}(\pm 1)$ of $\tilde{\varphi}$ have equal dimension n.

From the definition it follows that $\tilde{\varphi}\xi = 0$, $\eta \circ \tilde{\varphi} = 0$ and the endomorphism $\tilde{\varphi}$ has rank $2n$. When the tensor field $N_{\tilde{\varphi}} := [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric $\tilde{g}$ such that

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ such that $\tilde{g}(X_i, X_j) = \delta_{ij}$, $\tilde{g}(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \tilde{\varphi}X_i$, for any $i, j \in \{1, \dots, n\}$. Such basis is called a $\tilde{\varphi}$-basis.

If in addition $d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$ for all vector fields X, Y on M, $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde{h} := \frac{1}{2}\mathcal{L}_\xi \tilde{\varphi}$. It is known [31] that $\tilde{h}$ anti-commutes with $\tilde{\varphi}$ and satisfies tr$\tilde{h} = 0 = \tilde{h}\xi$ and

$$\tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}, \tag{2.2}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, \tilde{g})$. Moreover $\tilde{h} \equiv 0$ if and only if ξ is a Killing vector field and in this case $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3 (see [6]). Moreover, in any para-Sasakian manifold

$$\tilde{R}(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.3}$$

holds, but unlike contact metric geometry the condition (2.3) not necessarily implies that the manifold is para-Sasakian. Differentiating $\tilde{\nabla}_Y \xi = -\tilde{\varphi}Y + \tilde{\varphi}\tilde{h}Y$, we get

$$\tilde{R}(X, Y)\xi = -(\tilde{\nabla}_X \tilde{\varphi})Y + (\tilde{\nabla}_Y \tilde{\varphi})X + (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y \tilde{\varphi}\tilde{h})X. \tag{2.4}$$

In [31], Zamkovoy proved that

$$(\tilde{\nabla}_\xi \tilde{h})X = -\tilde{\varphi}X + \tilde{h}^2\tilde{\varphi}X + \tilde{\varphi}\tilde{R}(\xi, X)\xi. \tag{2.5}$$

Moreover, he showed that Ricci curvature $\tilde{S}$ in the direction of ξ is given by

$$\tilde{S}(\xi, \xi) = -2n + tr\tilde{h}^2. \tag{2.6}$$

An almost paracontact structure $(\tilde{\varphi}, \xi, \eta)$ is said to be *integrable* if $N_{\tilde{\varphi}}(X, Y) \in \Gamma(\mathbb{R}\xi)$ whenever $X, Y \in \Gamma(\mathcal{D})$.

In [27], J. Welyczko proved that any 3-dimensional paracontact metric manifold is always integrable. So for 3-dimensional paracontact metric manifold, we have

$$(\tilde{\nabla}_X \tilde{\varphi})Y = -\tilde{g}(X - \tilde{h}X, Y)\xi + \eta(Y)(X - \tilde{h}X). \tag{2.7}$$

We end this section by pointing out the following.

Theorem 2.1 ([11])**.** *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric (κ, μ)-space. Then M admits a canonical paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ given by*

$$\tilde{\varphi} := \frac{1}{\sqrt{1-\kappa}}h, \quad \tilde{g} := \frac{1}{\sqrt{1-\kappa}}d\eta(\cdot, h\cdot) + \eta \otimes \eta. \tag{2.8}$$

3. Preliminary results on 2n+1-dimensional paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifolds

Theorem 2.1 motivates the following definition.

Definition 3.1. *A $2n+1$-dimensional paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold is a paracontact metric manifold for which the curvature tensor field satisfies*

$$\tilde{R}(X, Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) + \tilde{\nu}(\eta(Y)\tilde{\varphi}\tilde{h}X - \eta(X)\tilde{\varphi}\tilde{h}Y), \tag{3.1}$$

for all $X, Y \in \Gamma(TM)$, where $\tilde{\kappa}, \tilde{\mu}, \tilde{\nu}$ are smooth functions on M.

In this section, we discuss some properties of paracontact metric manifolds satisfying the condition (3.1). We start with some preliminary properties.

Lemma 3.2. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a $2n+1$-dimensional paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold. Then the following identities hold:*

$$\tilde{h}^2 = (1+\tilde{\kappa})\tilde{\varphi}^2, \tag{3.2}$$

$$\tilde{Q}\xi = 2n\tilde{\kappa}\xi, \tag{3.3}$$

$$(\tilde{\nabla}_X\tilde{\varphi})Y = -\tilde{g}(X-\tilde{h}X,Y)\xi + \eta(Y)(X-\tilde{h}X), \text{ for } \tilde{\kappa}\neq -1, \tag{3.4}$$

$$\begin{aligned}(\tilde{\nabla}_X\tilde{h})Y - (\tilde{\nabla}_Y\tilde{h})X = &-(1+\tilde{\kappa})(2\tilde{g}(X,\tilde{\varphi}Y)\xi + \eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X)\\ &+(1-\tilde{\mu})(\eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X)\\ &-\tilde{\nu}(\eta(X)\tilde{h}Y - \eta(Y)\tilde{h}X),\end{aligned} \tag{3.5}$$

$$\begin{aligned}(\tilde{\nabla}_X\tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y\tilde{\varphi}\tilde{h})X = &-(1+\tilde{\kappa})(\eta(X)Y - \eta(Y)X)\\ &+(1-\tilde{\mu})(\eta(X)\tilde{h}Y - \eta(Y)\tilde{h}X)\\ &-\tilde{\nu}(\eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X),\end{aligned} \tag{3.6}$$

$$\tilde{\nabla}_\xi\tilde{h} = \tilde{\mu}\tilde{h}\circ\tilde{\varphi} - \tilde{\nu}\tilde{h}, \quad \tilde{\nabla}_\xi\tilde{\varphi}\tilde{h} = -\tilde{\mu}\tilde{h} + \tilde{\nu}\tilde{h}\circ\tilde{\varphi}, \tag{3.7}$$

$$\begin{aligned}\tilde{R}_{\xi X}Y = &\tilde{\kappa}(\tilde{g}(X,Y)\xi - \eta(Y)X) + \tilde{\mu}(\tilde{g}(\tilde{h}X,Y)\xi - \eta(Y)\tilde{h}X)\\ &+\tilde{\nu}(\tilde{g}(\tilde{\varphi}\tilde{h}X,Y)\xi - \eta(Y)\tilde{\varphi}\tilde{h}X),\end{aligned} \tag{3.8}$$

$$\xi(\tilde{\kappa}) = -2\tilde{\nu}(1+\tilde{\kappa}), \tag{3.9}$$

for any vector fields X, Y on M, where $\tilde{Q}$ denotes the Ricci operator of $(M,\tilde{g})$.

Proof. The proof of (3.2)-(3.5) are similar to that of [[12], Lemma 3.2]. The relation (3.6) is an immediate consequence of (3.4), (3.5) and $(\tilde{\nabla}_X\tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y\tilde{\varphi}\tilde{h})X = (\tilde{\nabla}_X\tilde{\varphi})\tilde{h}Y - (\tilde{\nabla}_Y\tilde{\varphi})\tilde{h}X + \tilde{\varphi}((\tilde{\nabla}_X\tilde{h})Y - (\tilde{\nabla}_Y\tilde{h})X)$. Putting ξ instead of X in the relation (3.5), we have (3.7). Taking into account $\tilde{h}\tilde{\varphi} = -\tilde{\varphi}\tilde{h}$, (3.2) and (3.7), we obtain

$$\tilde{\nabla}_\xi\tilde{h}^2 = (\tilde{\nabla}_\xi\tilde{h})\tilde{h} + \tilde{h}(\tilde{\nabla}_\xi\tilde{h}) = (\tilde{\mu}\tilde{h}\tilde{\varphi} - \tilde{\nu}\tilde{h})\tilde{h} + \tilde{h}(\tilde{\mu}\tilde{h}\tilde{\varphi} - \tilde{\nu}\tilde{h}) = -2\tilde{\nu}(1+\tilde{\kappa})\tilde{\varphi}^2. \tag{3.10}$$

Alternately, differentiating (3.2) along ξ and using (2.7), we obtain

$$\tilde{\nabla}_\xi\tilde{h}^2 = \xi(\tilde{\kappa})\tilde{\varphi}^2. \tag{3.11}$$

Combining (3.10) and (3.11), we complete the proof (3.9). □

Remarkable subclasses of paracontact $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifolds are given, in view of (2.3), by para-Sasakian manifolds, and by those paracontact metric manifolds such that $\tilde{R}(X,Y)\xi = 0$ for all vector fields X,Y on M. Note that, by (3.2), a paracontact $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold such that $\tilde{\kappa} = -1$ satisfies $\tilde{h}^2 = 0$. Unlike the contact metric case, since the metric $\tilde{g}$ is pseudo-Riemannian we can not conclude that $\tilde{h}$ vanishes and so the manifold is para-Sasakian.

Given a paracontact metric structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$ and $\alpha > 0$, the change of structure tensors

$$\bar{\eta} = \alpha\eta, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \bar{\varphi} = \tilde{\varphi}, \quad \bar{g} = \alpha\tilde{g} + \alpha(\alpha-1)\eta\otimes\eta \tag{3.12}$$

is called a *$\mathcal{D}_\alpha$-homothetic deformation.* One can easily check that the new structure $(\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ is still a paracontact metric structure [31].

Proposition 3.3 ([12])**.** *Let $(\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ be a paracontact metric structure obtained from $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ by a $\mathcal{D}_\alpha$-homothetic deformation. Then we have the following relationship between the Levi-Civita connections*

$\bar{\nabla}$ and $\tilde{\nabla}$ of $\bar{g}$ and $\tilde{g}$, respectively,

$$\bar{\nabla}_X Y = \tilde{\nabla}_X Y + \frac{\alpha-1}{\alpha}\tilde{g}(\tilde{\varphi}\tilde{h}X,Y)\xi - (\alpha-1)\left(\eta(Y)\tilde{\varphi}X + \eta(X)\tilde{\varphi}Y\right). \tag{3.13}$$

Furthermore,

$$\bar{h} = \frac{1}{\alpha}\tilde{h}. \tag{3.14}$$

After some straightforward calculations one can prove the following proposition.

Proposition 3.4 ([12]). *Under the same assumptions of Proposition 3.3, the curvature tensor fields $\bar{R}$ and $\tilde{R}$ are related by*

$$\begin{aligned}\alpha\bar{R}(X,Y)\bar{\xi} = {}& \tilde{R}(X,Y)\xi - (\alpha-1)((\tilde{\nabla}_X\tilde{\varphi})Y - (\tilde{\nabla}_Y\tilde{\varphi})X + \eta(Y)(X-\tilde{h}X) - \eta(X)(Y-\tilde{h}Y)) \\ & - (\alpha-1)^2(\eta(Y)X - \eta(X)Y).\end{aligned} \tag{3.15}$$

Using Proposition 3.4 one can easily get following

Proposition 3.5. *If $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ is a paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold, then $(\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ is a paracontact metric $(\bar{\kappa},\bar{\mu},\bar{\nu})$-structure, where*

$$\bar{\kappa} = \frac{\tilde{\kappa}+1-\alpha^2}{\alpha^2}, \quad \bar{\mu} = \frac{\tilde{\mu}+2\alpha-2}{\alpha}, \quad \bar{\nu} = \frac{\tilde{\nu}}{\alpha}\ . \tag{3.16}$$

By the same argument in Proposition 3.9 of [12], we have

Corollary 3.6. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold such that $\tilde{\kappa}\neq -1$. Then the operator $\tilde{h}$ in the case $\tilde{\kappa} > -1$ and the operator $\tilde{\varphi}\tilde{h}$ in the case $\tilde{\kappa} < -1$ are diagonalizable and admit three eigenvalues: 0, associate with the eigenvector ξ, $\tilde{\lambda}$ and -$\tilde{\lambda}$, of multiplicity n, where $\tilde{\lambda} := \sqrt{|1+\tilde{\kappa}|}$. The corresponding eigendistributions $\mathcal{D}_{\tilde{h}}(0) = \mathbb{R}\xi$, $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(0) = \mathbb{R}\xi$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ are mutually orthogonal and one has $\tilde{\varphi}\mathcal{D}_{\tilde{h}}(\tilde{\lambda}) = \mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$, $\tilde{\varphi}\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}) = \mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\tilde{\varphi}\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}) = \mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$, $\tilde{\varphi}\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}) = \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$. Furthermore,*

$$\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \left\{X \pm \frac{1}{\sqrt{1+\tilde{\kappa}}}\tilde{h}X | X \in \Gamma(\mathcal{D}^{\mp})\right\} \tag{3.17}$$

in the case $\tilde{\kappa} > -1$, and

$$\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}) = \left\{X \pm \frac{1}{\sqrt{-1-\tilde{\kappa}}}\tilde{\varphi}\tilde{h}X | X \in \Gamma(\mathcal{D}^{\mp})\right\} \tag{3.18}$$

where $\mathcal{D}^{+}$ and $\mathcal{D}^{-}$ denote the eigendistributions of $\tilde{\varphi}$ corresponding to the eigenvalues corresponding to the eigenvalues 1 and -1, respectively.

In the sequel, unless otherwise stated, we will always assume the index of $\mathcal{D}_{\tilde{h}}(\pm\lambda)$ (in the case $\tilde{\kappa} > -1$) and of $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\lambda)$ (in the case $\tilde{\kappa} < -1$) to be constant.

Being $\tilde{h}$ (in the case $\tilde{\kappa} > -1$) or $\tilde{\varphi}\tilde{h}$ (in the case $\tilde{\kappa} < -1$) diagonalizable, one can easily prove the following lemma. Following similar steps in proof of the theorem ([12], Lemma 3.11), we can give following lemma.

Lemma 3.7. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold such that $\tilde{\kappa}\neq -1$. If $\tilde{\kappa} > -1$ (respectively, $\tilde{\kappa} < -1$), then there exists a local orthogonal $\tilde{\varphi}$-basis $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,\xi\}$ of eigenvectors of $\tilde{h}$ (respectively, $\tilde{\varphi}\tilde{h}$) such that $X_1,\ldots,X_n \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ (respectively, $\Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$), $Y_1,\ldots,Y_n \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ (respectively, $\Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$), and*

$$\tilde{g}(X_i,X_i) = -\tilde{g}(Y_i,Y_i) = \begin{cases} 1, & \text{for } 1\leq i\leq r \\ -1, & \text{for } r+1\leq i\leq r+s \end{cases} \tag{3.19}$$

where $r = \operatorname{index}(\mathcal{D}_{\tilde h}(-\tilde\lambda))$ *(respectively,* $r = \operatorname{index}(\mathcal{D}_{\tilde\varphi\tilde h}(-\tilde\lambda))$*) and* $s = n-r = \operatorname{index}(\mathcal{D}_{\tilde h}(\tilde\lambda))$ *(respectively,* $s = \operatorname{index}(\mathcal{D}_{\tilde\varphi\tilde h}(\tilde\lambda))$*).*

Lemma 3.8. *The following differential equation is satisfied on every* $(2n+1)$*-dimensional paracontact metric* $(\tilde\kappa,\tilde\mu,\tilde\nu)$*-manifold* $(M,\tilde\varphi,\xi,\eta,\tilde g)$ *:*

$$\begin{aligned}0 &= \xi(\tilde\kappa)(\eta(Y)X-\eta(X)Y)+\xi(\tilde\mu)(\eta(Y)\tilde hX-\eta(X)\tilde hY)+\xi(\tilde\nu)(\eta(Y)\tilde\varphi\tilde hX-\eta(X)\tilde\varphi\tilde hY)\\&\quad+X(\tilde\kappa)\tilde\varphi^2Y-Y(\tilde\kappa)\tilde\varphi^2X+X(\tilde\mu)\tilde hY-Y(\tilde\mu)\tilde hX+X(\tilde\nu)\tilde\varphi\tilde hY-Y(\tilde\nu)\tilde\varphi\tilde hX.\end{aligned}\tag{3.20}$$

Proof. Differentiating (3.1) along an arbitary vector field Z and using the relation $\tilde\nabla\xi=-\tilde\varphi+\tilde\varphi\tilde h$ we find

$$\begin{aligned}\tilde\nabla_Z\tilde R(X,Y)\xi &= Z(\tilde\kappa)(\eta(Y)X-\eta(X)Y)+Z(\tilde\mu)(\eta(Y)\tilde hX-\eta(X)\tilde hY)+Z(\tilde\nu)(\eta(Y)\tilde\varphi\tilde hX-\eta(X)\tilde\varphi\tilde hY)\\&\quad+\tilde\kappa[(\eta(\tilde\nabla_ZY)-\tilde g(Y,\tilde\varphi Z)+\tilde g(Y,\tilde\varphi\tilde hZ))X+\eta(Y)\tilde\nabla_ZX\\&\quad+(-\eta(\tilde\nabla_ZX)+\tilde g(X,\tilde\varphi Z)-\tilde g(X,\tilde\varphi\tilde hZ))Y-\eta(X)\tilde\nabla_ZY]\\&\quad+\tilde\mu[(\eta(\tilde\nabla_ZY)-\tilde g(Y,\tilde\varphi Z)+\tilde g(Y,\tilde\varphi\tilde hZ))\tilde hX+\eta(Y)\tilde\nabla_Z\tilde hX\\&\quad+(-\eta(\tilde\nabla_ZX)+\tilde g(X,\tilde\varphi Z)-\tilde g(X,\tilde\varphi\tilde hZ))\tilde hY-\eta(X)\tilde\nabla_Z\tilde hY]\\&\quad+\tilde\nu[(\eta(\tilde\nabla_ZY)-\tilde g(Y,\tilde\varphi Z)+\tilde g(Y,\tilde\varphi\tilde hZ))\tilde\varphi\tilde hX+\eta(Y)\tilde\nabla_Z\tilde\varphi\tilde hX\\&\quad+(-\eta(\tilde\nabla_ZX)+\tilde g(X,\tilde\varphi Z)-\tilde g(X,\tilde\varphi\tilde hZ))\tilde\varphi\tilde hY-\eta(X)\tilde\nabla_Z\tilde\varphi\tilde hY].\end{aligned}$$

By using the relation $\tilde\nabla\xi=-\tilde\varphi+\tilde\varphi\tilde h$, we deduce

$$\begin{aligned}(\tilde\nabla_Z\tilde R)(X,Y,\xi) &= \tilde\nabla_Z\tilde R(X,Y)\xi-\tilde R(\tilde\nabla_ZX,Y)\xi-\tilde R(X,\tilde\nabla_ZY)\xi-\tilde R(X,Y)\tilde\nabla_Z\xi\\&= Z(\tilde\kappa)(\eta(Y)X-\eta(X)Y)+Z(\tilde\mu)(\eta(Y)\tilde hX-\eta(X)\tilde hY)+Z(\tilde\nu)(\eta(Y)\tilde\varphi\tilde hX-\eta(X)\tilde\varphi\tilde hY)\\&\quad+\tilde\kappa[\tilde g(Y,-\tilde\varphi Z+\tilde\varphi\tilde hZ)X+\tilde g(X,\tilde\varphi Z-\tilde\varphi\tilde hZ)Y]\\&\quad+\tilde\mu[\tilde g(Y,-\tilde\varphi Z+\tilde\varphi\tilde hZ)\tilde hX+\tilde g(X,\tilde\varphi Z-\tilde\varphi\tilde hZ)\tilde hY\\&\quad+\eta(Y)(\tilde\nabla_Z\tilde h)X-\eta(X)(\tilde\nabla_Z\tilde h)Y]\\&\quad+\tilde\nu[\tilde g(Y,-\tilde\varphi Z+\tilde\varphi\tilde hZ)\tilde\varphi\tilde hX+\tilde g(X,\tilde\varphi Z-\tilde\varphi\tilde hZ))\tilde\varphi\tilde hY\\&\quad+\eta(Y)(\tilde\nabla_Z\tilde\varphi\tilde h)X-\eta(X)(\tilde\nabla_Z\tilde\varphi\tilde h)Y]+\tilde R(X,Y)\tilde\varphi Z-\tilde R(X,Y)\tilde\varphi\tilde hZ.\end{aligned}$$

Using the second Bianchi identity in the last relation, we obtain

$$\begin{aligned}0 &= Z(\tilde\kappa)(\eta(Y)X-\eta(X)Y)+Z(\tilde\mu)(\eta(Y)\tilde hX-\eta(X)\tilde hY)+Z(\tilde\nu)(\eta(Y)\tilde\varphi\tilde hX-\eta(X)\tilde\varphi\tilde hY)\\&\quad+X(\tilde\kappa)(\eta(Z)Y-\eta(Y)Z)+X(\tilde\mu)(\eta(Z)\tilde hY-\eta(Y)\tilde hZ)+X(\tilde\nu)(\eta(Z)\tilde\varphi\tilde hY-\eta(Y)\tilde\varphi\tilde hZ)\\&\quad+Y(\tilde\kappa)(\eta(X)Z-\eta(Z)X)+Y(\tilde\mu)(\eta(X)\tilde hZ-\eta(Z)\tilde hX)+Y(\tilde\nu)(\eta(X)\tilde\varphi\tilde hZ-\eta(Z)\tilde\varphi\tilde hX)\\&\quad+2\tilde\kappa[\tilde g(\tilde\varphi Y,Z)X+\tilde g(\tilde\varphi Z,X)Y+\tilde g(\tilde\varphi X,Y)Z]\\&\quad+\tilde\mu[2\tilde g(\tilde\varphi Y,Z)\tilde hX+\eta(Z)((\tilde\nabla_X\tilde h)Y-(\tilde\nabla_Y\tilde h)X)+2\tilde g(\tilde\varphi Z,X)\tilde hY+\eta(X)((\tilde\nabla_Y\tilde h)Z-(\tilde\nabla_Z\tilde h)Y)\\&\quad+2\tilde g(\tilde\varphi X,Y)\tilde hZ+\eta(Y)((\tilde\nabla_Z\tilde h)X-(\tilde\nabla_X\tilde h)Z)]\\&\quad+\tilde\nu[2\tilde g(\tilde\varphi Y,Z)\tilde\varphi\tilde hX+\eta(Z)((\tilde\nabla_X\tilde\varphi\tilde h)Y-(\tilde\nabla_Y\tilde\varphi\tilde h)X)+2\tilde g(\tilde\varphi Z,X)\tilde\varphi\tilde hY+\eta(X)((\tilde\nabla_Y\tilde\varphi\tilde h)Z-(\tilde\nabla_Z\tilde\varphi\tilde h)Y)\\&\quad+2\tilde g(\tilde\varphi X,Y)\tilde\varphi\tilde hZ+\eta(Y)((\tilde\nabla_Z\tilde\varphi\tilde h)X-(\tilde\nabla_X\tilde\varphi\tilde h)Z)]\\&\quad+\tilde R(X,Y)\tilde\varphi Z+\tilde R(Y,Z)\tilde\varphi X+\tilde R(Z,X)\tilde\varphi Y\\&\quad-\tilde R(X,Y)\tilde\varphi\tilde hZ-\tilde R(Y,Z)\tilde\varphi\tilde hX-\tilde R(Z,X)\tilde\varphi\tilde hY.\end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. Putting ξ instead of Z in the last relation, we obtain

$$\begin{aligned} 0 &= \xi(\tilde{\kappa})(\eta(Y)X - \eta(X)Y) + \xi(\tilde{\mu})(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) + \xi(\tilde{\nu})(\eta(Y)\tilde{\varphi}\tilde{h}X - \eta(X)\tilde{\varphi}\tilde{h}Y) \\ &\quad + X(\tilde{\kappa})Y - \eta(Y)\xi + X(\tilde{\mu})\tilde{h}Y + X(\tilde{\nu})\tilde{\varphi}\tilde{h}Y \\ &\quad + Y(\tilde{\kappa})(\eta(X)\xi - X) - Y(\tilde{\mu})\tilde{h}X - Y(\tilde{\nu})\tilde{\varphi}\tilde{h}X + 2\tilde{\kappa}\tilde{g}(\tilde{\varphi}X, Y)\xi \\ &\quad + \tilde{\mu}[(\tilde{\nabla}_X\tilde{h})Y - (\tilde{\nabla}_Y\tilde{h})X + \eta(X)((\tilde{\nabla}_Y\tilde{h})\xi - (\tilde{\nabla}_\xi\tilde{h})Y) + \eta(Y)((\tilde{\nabla}_\xi\tilde{h})X - (\tilde{\nabla}_X\tilde{h})\xi)] \\ &\quad + \tilde{\nu}[(\tilde{\nabla}_X\tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y\tilde{\varphi}\tilde{h})X + \eta(X)((\tilde{\nabla}_Y\tilde{\varphi}\tilde{h})\xi - (\tilde{\nabla}_\xi\tilde{\varphi}\tilde{h})Y) \\ &\quad + \eta(Y)((\tilde{\nabla}_\xi\tilde{\varphi}\tilde{h})X - (\tilde{\nabla}_X\tilde{\varphi}\tilde{h})\xi)] \\ &\quad + \tilde{R}(Y,\xi)\tilde{\varphi}X + \tilde{R}(\xi,X)\tilde{\varphi}Y - \tilde{R}(Y,\xi)\tilde{\varphi}\tilde{h}X - \tilde{R}(\xi,X)\tilde{\varphi}\tilde{h}Y. \end{aligned}$$

Using (3.5),(3.6) and (3.8) in the last relation we finally get (3.20) and it completes the proof. □

By Corollary 3.6 and Lemma 3.7, one can obtain the proof of following theorem by using a similar method of ([20], Theorem 4.1).

Theorem 3.9. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a $2n+1$-dimensional paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold with $\tilde{\kappa} \neq -1$ and $n > 1$. Then M is a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-manifold, i.e $\tilde{\kappa}, \tilde{\mu}$ are constants and $\tilde{\nu}$ is the zero function.*

Remark 3.10. By (3.2), for $\tilde{\kappa} = -1$ we obtain $\tilde{h}^2 = 0$. Since the metric $\tilde{g}$ is not positive definite we can not conclude that $\tilde{h} = 0$ and the manifold is para-Sasakian. The following question comes up naturally. Do there exist a paracontact metric $(-1, \tilde{\mu})$ manifold with dimension greater than 3 satisfying $\tilde{h}^2 = 0$ but $\tilde{h} \neq 0$? B. Cappelletti Montano and L. Di Terlizzi [11] and V. Martin-Molina [23] gave an affirmative answer for $n > 1$.

In (κ, μ, ν)-contact metric case, T.Koufogiorgos et al. [20] proved that if dimension greater than 3 then it is either a Sasakian manifold or a (κ, μ)-contact metric manifold. Now we will give an example of paracontact metric $(-1, \tilde{\mu}, \tilde{\nu} \neq 0)$-manifold for five dimensional which have no contact metric counterpart ($\tilde{h}^2 = 0$ but $\tilde{h} \neq 0$). This is first example of this type manifold .

Example 3.11. *Let $\mathfrak{g}$ be the 5-dimensional Lie algebra with basis X_1, X_2, Y_1, Y_2, ξ and Lie brackets defined by*

$$\begin{array}{cc} [\xi, X_1] = Y_1\ , & [\xi, Y_1] = 0 \\ [\xi, X_2] = Y_2\ , & [\xi, X_2] = Y_2 \\ [X_1, Y_1] = 2\xi - Y_1\ , & [X_2, Y_2] = \frac{2}{3}Y_1 - Y_2 - 2\xi \\ [X_1, X_2] = -\frac{1}{3}X_1 - \frac{1}{3}X_2, & [Y_1, Y_2] = 0 \\ [X_1, Y_2] = -\frac{2}{3}Y_1 - \frac{1}{3}Y_2, & [Y_1, X_2] = \frac{1}{3}Y_1. \end{array}$$

Let $\mathcal{G}$ be a Lie group whose Lie algebra is $\mathfrak{g}$. On $\mathcal{G}$ we define a left-invariant paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ by setting $\tilde{\varphi}\xi = 0$ and $\tilde{\varphi}X_i = X_i$, $\tilde{\varphi}Y_i = -Y_i, \eta(\xi) = 1, \eta(X_i) = \eta(Y_i) = 0$ and whose only non-vanishing components of the metric are $\tilde{g}(\xi,\xi) = \tilde{g}(X_1, Y_1) = 1$, $\tilde{g}(X_2, Y_2) = -1$ for all $i = 1, 2$. Therefore, $\tilde{h}X_i = Y_i$ and $\tilde{h}Y_i = 0$, $i = 1, 2$, so $\tilde{h}^2 = 0$ but $\tilde{h} \neq 0$ and rank$(\tilde{h})$= 2.Moreover, one can verify that $(\mathcal{G}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is a paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold with $\tilde{\kappa} = -1, \tilde{\mu} = -1, \tilde{\nu} = -3$.

4. Classification of the 3-dimensional Paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifolds

In this section, we analyze the different possibilities for the tensor field $\tilde{h}$. Hence one can recognize the differences between the contact and paracontact cases by looking at the possible Jordan forms of the tensor field $\tilde{h}$.

A self-adjoint linear operator A of a Riemannian manifold is always diagonalizable, but this is not the case for a self-adjoint linear operator A of a Lorentzian manifold. It is known ([26], pp. 50-55) that self-adjoint linear operator of a vector space with a Lorentzian inner product can be put into four possible canonical forms. In particular, the matrix representation $\tilde{g}$ of the induced metric on M_1^3 is of Lorentz type, so the self-adjoint linear A of M_1^3 can be put into one of the following four forms with respect to frames $\{e_1, e_2, e_3\}$ at $T_pM_1^3$ where $T_pM_1^3$ is called tangent space to M at p [21],[24].

$$\text{(I)} \qquad A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \tilde{g} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{(II)} \qquad A = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \tilde{g} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{(III)} \qquad A = \begin{pmatrix} \gamma & -\lambda & 0 \\ \lambda & \gamma & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \tilde{g} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda \neq 0,$$

$$\text{(IV)} \qquad A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & 0 & \lambda \end{pmatrix}, \qquad \tilde{g} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrices $\tilde{g}$ for cases (I) and (III) are with respect to an orthonormal basis of $T_pM_1^3$, whereas for cases (II) and (IV) are with respect to a pseudo-orthonormal basis. This is a basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$ satisfying $\tilde{g}(e_1, e_1) = \tilde{g}(e_2, e_2) = \tilde{g}(e_1, e_3) = \tilde{g}(e_2, e_3) = 0$ and $\tilde{g}(e_1, e_2) = \tilde{g}(e_3, e_3) = 1$.

Next, we recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$\tilde{R}(X,Y)Z = \tilde{g}(Y,Z)\tilde{Q}X - \tilde{g}(X,Z)\tilde{Q}Y + \tilde{g}(\tilde{Q}Y,Z)X - \tilde{g}(\tilde{Q}X,Z)Y - \frac{r}{2}(\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y). \tag{4.1}$$

The tensor $\tilde{h}$ has the canonical form (I). Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact metric manifold .

$$\begin{aligned} U_1 &= \Big\{p \in M \mid \tilde{h}(p) \neq 0\Big\} \subset M \\ U_2 &= \Big\{p \in M \mid \tilde{h}(p) = 0, \text{in a neighborhood of p}\Big\} \subset M \end{aligned}$$

That $\tilde{h}$ is a smooth function on M implies $U_1 \cup U_2$ is an open and dense subset of M, so any property satisfied in $U_1 \cup U_2$ is also satisfied in M. For any point $p \in U_1 \cup U_2$ there exists a local orthonormal $\tilde{\varphi}$-basis $\{\tilde{e}, \tilde{\varphi}\tilde{e}, \xi\}$ of smooth eigenvectors of $\tilde{h}$ in a neighborhood of p, where $-\tilde{g}(\tilde{e}, \tilde{e}) = \tilde{g}(\tilde{\varphi}\tilde{e}, \tilde{\varphi}\tilde{e}) = \tilde{g}(\xi, \xi) = 1$. On U_1 we put $\tilde{h}\tilde{e} = \tilde{\lambda}\tilde{e}$, where $\tilde{\lambda}$ is a non-vanishing smooth function. Since $tr\tilde{h} = 0$, we have $\tilde{h}\tilde{\varphi}\tilde{e} = -\tilde{\lambda}\tilde{\varphi}\tilde{e}$. The eigenvalue function $\tilde{\lambda}$ is continuos on M and smooth on $U_1 \cup U_2$. So, $\tilde{h}$ has following form

$$\begin{pmatrix} \tilde{\lambda} & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

respect to local orthonormal $\tilde{\varphi}$-basis $\{\tilde{e}, \tilde{\varphi}\tilde{e}, \xi\}$. In this case, we will say the operator $\tilde{h}$ is of $\mathfrak{h}_1$ *type*. Using same method with [20] and [25], we have

Lemma 4.1. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold with $\tilde{h}$ of $\mathfrak{h}_1$ type. Then for the covariant derivative on U_1 the following equations are valid*

$$
\begin{aligned}
i)\ \tilde{\nabla}_{\tilde{e}}\tilde{e} &= \frac{1}{2\tilde{\lambda}}\left[\tilde{\sigma}(\tilde{e})-(\tilde{\varphi}\tilde{e})(\tilde{\lambda})\right]\tilde{\varphi}\tilde{e}, \quad ii)\ \tilde{\nabla}_{\tilde{e}}\tilde{\varphi}\tilde{e}=\frac{1}{2\tilde{\lambda}}\left[\tilde{\sigma}(\tilde{e})-(\tilde{\varphi}\tilde{e})(\tilde{\lambda})\right]\tilde{e}+(1-\tilde{\lambda})\xi,\\
iii)\ \tilde{\nabla}_{\tilde{e}}\xi &= (\tilde{\lambda}-1)\tilde{\varphi}\tilde{e},\\
iv)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{e} &= -\frac{1}{2\tilde{\lambda}}\left[\tilde{\sigma}(\tilde{\varphi}\tilde{e})+\tilde{e}(\tilde{\lambda})\right]\tilde{\varphi}\tilde{e}-(\tilde{\lambda}+1)\xi, \quad v)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{e}=-\frac{1}{2\tilde{\lambda}}\left[\tilde{\sigma}(\tilde{\varphi}\tilde{e})+\tilde{e}(\tilde{\lambda})\right]\tilde{e},\\
vi)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\xi &= -(\tilde{\lambda}+1)\tilde{e},\\
vii)\ \tilde{\nabla}_{\xi}\tilde{e} &= b\tilde{\varphi}\tilde{e}, \quad viii)\ \tilde{\nabla}_{\xi}\tilde{\varphi}\tilde{e}=b\tilde{e},\\
ix)\ [\tilde{e},\xi] &= (\tilde{\lambda}-1-b)\tilde{\varphi}\tilde{e}, \quad x)\ [\tilde{\varphi}\tilde{e},\xi]=(-\tilde{\lambda}-1-b)\tilde{e},\\
xi)\ [\tilde{e},\tilde{\varphi}\tilde{e}] &= \frac{1}{2\tilde{\lambda}}\left[\tilde{\sigma}(\tilde{e})-(\tilde{\varphi}\tilde{e})(\tilde{\lambda})\right]\tilde{e}+\frac{1}{2\tilde{\lambda}}\left[\tilde{\sigma}(\tilde{\varphi}\tilde{e})+\tilde{e}(\lambda)\right]\tilde{\varphi}\tilde{e}+2\xi,
\end{aligned}
\tag{4.3}
$$

where

$$b=\tilde{g}(\ \tilde{\nabla}_{\xi}\tilde{e},\tilde{\varphi}\tilde{e}),\ \tilde{\sigma}=\tilde{S}(\xi,.)_{\ker\eta}\ .$$

Proposition 4.2. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold. If $\tilde{h}$ is $\mathfrak{h}_1$ type then on U_1 we have*

$$\tilde{\nabla}_{\xi}\tilde{h}=-2b\tilde{h}\tilde{\varphi}+\xi(\tilde{\lambda})s, \tag{4.4}$$

where s is the $(1,1)$-type tensor defined by $s\xi=0$, $s\tilde{e}=\tilde{e}$, $s\tilde{\varphi}\tilde{e}=-\tilde{\varphi}\tilde{e}$.

Proof. Using (4.3), we get

$$
\begin{aligned}
(\tilde{\nabla}_{\xi}\tilde{h})\xi &= 0=(-2b\tilde{h}\tilde{\varphi}+\xi(\tilde{\lambda})s)\xi,\\
(\tilde{\nabla}_{\xi}\tilde{h})\tilde{e} &= 2\tilde{\lambda}b\tilde{\varphi}\tilde{e}+\xi(\tilde{\lambda})\tilde{e}=(-2b\tilde{h}\tilde{\varphi}+\xi(\tilde{\lambda})s)\tilde{e},\\
(\tilde{\nabla}_{\xi}\tilde{h})\tilde{\varphi}\tilde{e} &= -2\tilde{\lambda}b\tilde{e}-\xi(\tilde{\lambda})\tilde{\varphi}\tilde{e}=(-2b\tilde{h}\tilde{\varphi}+\xi(\tilde{\lambda})s)\tilde{\varphi}\tilde{e}.
\end{aligned}
$$

So, the last equations complete the proof of (4.4). □

Proposition 4.3. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold with $\tilde{h}$ of $\mathfrak{h}_1$ type. Then the following equation holds on M.*

$$\tilde{h}^2-\tilde{\varphi}^2=\frac{\tilde{S}(\xi,\xi)}{2}\tilde{\varphi}^2. \tag{4.5}$$

Proof. Using (2.6), we have $\tilde{S}(\xi,\xi)=2(\tilde{\lambda}^2-1)$. After calculating $\tilde{h}^2-\tilde{\varphi}^2$ with respect to the basis components, we get

$$\tilde{h}^2\xi-\tilde{\varphi}^2\xi=\frac{\tilde{S}(\xi,\xi)}{2}\tilde{\varphi}^2\xi=0,\ \ \tilde{h}^2\tilde{e}-\tilde{\varphi}^2\tilde{e}=\frac{\tilde{S}(\xi,\xi)}{2}\tilde{\varphi}^2\tilde{e},\ \ \tilde{h}^2\tilde{\varphi}\tilde{e}-\tilde{\varphi}^3\tilde{e}=\frac{\tilde{S}(\xi,\xi)}{2}\tilde{\varphi}^2\tilde{\varphi}\tilde{e}. \tag{4.6}$$

Thus the last equation completes the proof of (4.5). □

Lemma 4.4. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold. If $\tilde{h}$ is $\mathfrak{h}_1$ type then the Ricci operator $\tilde{Q}$ is given by*

$$\tilde{Q}=a_1I+b_1\eta\otimes\xi-\tilde{\varphi}(\tilde{\nabla}_{\xi}\tilde{h})+\tilde{\sigma}(\tilde{\varphi}^2)\otimes\xi-\tilde{\sigma}(\tilde{e})\eta\otimes\tilde{e}+\tilde{\sigma}(\tilde{\varphi}\tilde{e})\eta\otimes\tilde{\varphi}\tilde{e}, \tag{4.7}$$

where a_1 and b_1 are smooth functions defined by $a_1=1-\tilde{\lambda}^2+\frac{r}{2}$ and $b_1=3(\tilde{\lambda}^2-1)-\frac{r}{2}$, respectively. Moreover the components of the Ricci operator $\tilde{Q}$ are given by

$$
\begin{aligned}
\tilde{Q}\xi &= (a_1+b_1)\xi - \tilde{\sigma}(\tilde{e})\tilde{e} + \tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e},\\
\tilde{Q}\tilde{e} &= \tilde{\sigma}(\tilde{e})\xi + (a_1 - 2b\tilde{\lambda})\tilde{e} - \xi(\tilde{\lambda})\tilde{\varphi}\tilde{e},\\
\tilde{Q}\tilde{\varphi}\tilde{e} &= \tilde{\sigma}(\tilde{\varphi}\tilde{e})\xi + \xi(\tilde{\lambda})\tilde{e} + (a_1 + 2b\tilde{\lambda})\tilde{\varphi}\tilde{e}.
\end{aligned} \tag{4.8}
$$

Proof. From (4.1), we have

$$\tilde{l}X = \tilde{R}(X,\xi)\xi = \tilde{S}(\xi,\xi)X - \tilde{S}(X,\xi)\xi + \tilde{Q}X - \eta(X)\tilde{Q}\xi - \frac{r}{2}(X - \eta(X)\xi),$$

where $\tilde{l}$ denotes the Jacobi operator and X is a vector field. Using (2.5), the last equation implies

$$\tilde{Q}X = -\tilde{\varphi}^2X + \tilde{h}^2X - \tilde{\varphi}(\tilde{\nabla}_\xi\tilde{h})X - \tilde{S}(\xi,\xi)X + \tilde{S}(X,\xi)\xi + \eta(X)\tilde{Q}\xi + \frac{r}{2}(X - \eta(X)\xi).$$

Since $\tilde{S}(X,\xi) = \tilde{S}(\tilde{\varphi}^2X,\xi) + \eta(X)\tilde{S}(\xi,\xi)$, we have

$$\tilde{Q}X = \frac{\tilde{S}(\xi,\xi)}{2}\tilde{\varphi}^2X - \tilde{\varphi}(\tilde{\nabla}_\xi\tilde{h})X - \tilde{S}(\xi,\xi)X + \tilde{S}(\tilde{\varphi}^2X,\xi)\xi + \eta(X)\tilde{S}(\xi,\xi)\xi + \eta(X)\tilde{Q}\xi + \frac{r}{2}\tilde{\varphi}^2X. \tag{4.9}$$

One can easily prove that

$$\tilde{Q}\xi = -\tilde{\sigma}(\tilde{e})\tilde{e} + \tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e} + \tilde{S}(\xi,\xi)\xi. \tag{4.10}$$

Using (4.10) in (4.9), we have

$$
\begin{aligned}
\tilde{Q}X \;=\; &\left(1 - \tilde{\lambda}^2 + \frac{r}{2}\right)X + \left(3(\tilde{\lambda}^2 - 1) - \frac{r}{2}\right)\eta(X)\xi\\
&-\tilde{\varphi}(\tilde{\nabla}_\xi\tilde{h})X + \tilde{\sigma}(\tilde{\varphi}^2X)\xi - \eta(X)\tilde{\sigma}(\tilde{e})\tilde{e} + \eta(X)\tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e},
\end{aligned} \tag{4.11}
$$

for arbitrary vector field X. Hence, the proof follows from (4.11). By (4.4) and (4.11) we get (4.8). □

The tensor $\tilde{h}$ has the canonical form (II). Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold and p is a point of M. Then there exists a local pseudo-orthonormal basis $\{e_1,e_2,\xi\}$ in a neighborhood of p where $\tilde{g}(e_1,e_1) = \tilde{g}(e_2,e_2) = \tilde{g}(e_1,\xi) = \tilde{g}(e_2,\xi) = 0$ and $\tilde{g}(e_1,e_2) = 1$.

Lemma 4.5. *Let $\mathcal{U}$ be the open subset of M where $\tilde{h} \neq 0$. For every $p \in \mathcal{U}$ there exists an open neighborhood of p such that $\tilde{h}e_1 = e_2, \tilde{h}e_2 = 0, \tilde{h}\xi = 0$ and $\tilde{\varphi}e_1 = \pm e_1$, $\tilde{\varphi}e_2 = \mp e_2$.*

Proof. Since the tensor $\tilde{h}$ has canonical form (II) (with respect to a pseudo-orthonormal basis $\{e_1,e_2,\xi\}$) then $\tilde{h}e_1 = \tilde{\lambda}e_1 + e_2$, $\tilde{h}e_2 = \tilde{\lambda}e_2$, $\tilde{h}\xi = 0$. Since $\tilde{h}\xi = 0$ and $tr(h) = 0$ we have $\tilde{\lambda} = 0$. On the other hand using the anti-symmetry tensor field $\tilde{\varphi}$ of type $(1,1)$, with respect to the pseudo-orthonormal basis $\{e_1,e_2,\xi\}$, takes the form

$$\begin{pmatrix} \tilde{\varphi}_{11} & \tilde{\varphi}_{12} & 0\\ \tilde{\varphi}_{21} & \tilde{\varphi}_{22} & 0\\ 0 & 0 & 0 \end{pmatrix}. \tag{4.12}$$

Using (2.1) and (4.12) we have

$$
\begin{aligned}
\tilde{g}(\tilde{\varphi}e_1,e_1) &= 0 = \tilde{\varphi}_{21} \text{ and } \tilde{g}(\tilde{\varphi}e_2,e_2) = 0 = \tilde{\varphi}_{12},\\
\tilde{g}(\tilde{\varphi}e_1,e_2) &= \tilde{\varphi}_{11} = -\tilde{g}(e_1,\tilde{\varphi}e_2) = -\tilde{\varphi}_{22}
\end{aligned} \tag{4.13}
$$

Other hand we get

$$\tilde{g}(\tilde{\varphi}e_1,\tilde{\varphi}e_2) = \tilde{\varphi}_{11}\tilde{\varphi}_{22} = -g(e_1,e_2) = 1. \tag{4.14}$$

From two last equation , $\tilde{\varphi}_{22} = \mp 1$. This completes the proof. □

Hence the tensor $\tilde{h}$ has the form $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ relative a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$. In this case, we call $\tilde{h}$ is of $\mathfrak{h}_2$ *type*.

Remark 4.6. Without loss of generality, we can assume that $\tilde{\varphi}e_1 = e_1$ $\tilde{\varphi}e_2 = -e_2$. Moreover one can easily get $\tilde{h}^2 = 0$ but $\tilde{h} \neq 0$.

Lemma 4.7. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact metric manifold with $\tilde{h}$ of $\mathfrak{h}_2$ type. Then for the covariant derivative on $\mathcal{U}$ the following equations are valid*

$$\begin{aligned} i)\ \tilde{\nabla}_{e_1}e_1 &= -b_2e_1 + \xi, \quad ii)\ \tilde{\nabla}_{e_1}e_2 = b_2e_2 + \xi, \quad iii)\ \tilde{\nabla}_{e_1}\xi = -e_1 - e_2, \\ iv)\ \tilde{\nabla}_{e_2}e_1 &= -\tilde{b}_2e_1 - \xi, \quad v)\ \tilde{\nabla}_{e_2}e_2 = \tilde{b}_2e_2, \quad vi)\ \tilde{\nabla}_{e_2}\xi = e_2, \\ vii)\ \tilde{\nabla}_{\xi}e_1 &= a_2e_1, \quad viii)\ \tilde{\nabla}_{\xi}e_2 = -a_2e_2, \\ ix)\ [e_1, \xi] &= -(1 + a_2)e_1 - e_2, \quad x)\ [e_2, \xi] = (1 + a_2)e_2, \\ xi)\ [e_1, e_2] &= \tilde{b}_2e_1 + b_2e_2 + 2\xi. \end{aligned} \tag{4.15}$$

where $a_2 = \tilde{g}(\tilde{\nabla}_{\xi}e_1, e_2)$, $b_2 = \tilde{g}(\tilde{\nabla}_{e_1}e_2, e_1)$ and $\tilde{b}_2 = -\frac{1}{2}\ \tilde{\sigma}(e_1) = -\frac{1}{2}\tilde{S}(\xi, e_1)$.

Proof. Replacing X by e_1 and Y by e_2 in equation $\tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}$, we have $iii), vi)$.

For the proof of $viii)$ we have

$$\begin{aligned} \tilde{\nabla}_{\xi}e_2 &= \tilde{g}(\tilde{\nabla}_{\xi}e_2, e_2)e_1 + \tilde{g}(\tilde{\nabla}_{\xi}e_2, e_1)e_2 + \tilde{g}(\tilde{\nabla}_{\xi}e_2, \xi)\xi \\ &= -\tilde{g}(e_2, \tilde{\nabla}_{\xi}e_1)e_2. \end{aligned}$$

If the function a_2 is defined as $\tilde{g}(\tilde{\nabla}_{\xi}e_1, e_2)$ then $\tilde{\nabla}_{\xi}e_2 = -a_2e_2$. The proofs of other covariant derivative equalities are similar to $ii)$.

Putting $X = e_1$, $Y = e_2$ and $Z = \xi$ in the equation (4.1), we have

$$\tilde{R}(e_1, e_2)\xi = -\tilde{\sigma}(e_1)e_2 + \tilde{\sigma}(e_2)e_1. \tag{4.16}$$

On the other hand, by using (2.4), we get

$$\begin{aligned} \tilde{R}(e_1, e_2)\xi &= (\tilde{\nabla}_{e_1}\tilde{\varphi}\tilde{h})e_2 - (\tilde{\nabla}_{e_2}\tilde{\varphi}\tilde{h})e_1 \\ &= 2\tilde{b}_2e_2. \end{aligned} \tag{4.17}$$

Comparing (4.17) with (4.16), we obtain

$$\tilde{\sigma}(e_1) = -2\tilde{b}_2, \quad \tilde{\sigma}(e_2) = 0 = \tilde{S}(\xi, e_2). \tag{4.18}$$

Hence, the function $\tilde{b}_2$ is obtained from the last equation. □

Next, we derive a useful formula for $\tilde{\nabla}_{\xi}\tilde{h}$.

Proposition 4.8. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact metric manifold with $\tilde{h}$ of $\mathfrak{h}_2$ type. Then we have*

$$\tilde{\nabla}_{\xi}\tilde{h} = 2a_2\tilde{\varphi}\tilde{h}, \tag{4.19}$$

on $\mathcal{U}$.

Proof. Using (4.15), we get

$$\begin{aligned} (\tilde{\nabla}_{\xi}\tilde{h})\xi &= 0 = (2a_2\tilde{\varphi}\tilde{h})\xi, \\ (\tilde{\nabla}_{\xi}\tilde{h})e_1 &= -2a_2e_2 = (2a_2\tilde{\varphi}\tilde{h})e_1, \\ (\tilde{\nabla}_{\xi}\tilde{h})e_2 &= 0 = (2a_2\tilde{\varphi}\tilde{h})e_2. \end{aligned}$$

□

Using (2.6), we have $\tilde S(\xi,\xi)=-2$. After calculating $\tilde h^2-\tilde\varphi^2$ with respect to the basis components, we get

$$\tilde h^2\xi-\tilde\varphi^2\xi=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2\xi=0,\quad \tilde h^2e_1-\tilde\varphi^2e_1=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2e_1,\quad \tilde h^2e_2-\tilde\varphi^2e_2=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2e_2. \tag{4.20}$$

So we have following

Proposition 4.9. *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a 3-dimensional paracontact metric manifold. If $\tilde h$ is $\mathfrak{h}_2$ type then the following equation holds on M.*

$$\tilde h^2-\tilde\varphi^2=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2. \tag{4.21}$$

Lemma 4.10. *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a 3-dimensional paracontact metric manifold with $\tilde h$ of $\mathfrak{h}_2$ type. Then the Ricci operator $\tilde Q$ is given by*

$$\tilde Q=\ddot aI+\ddot b\eta\otimes\xi-\tilde\varphi(\tilde\nabla_\xi\tilde h)+\tilde\sigma(\tilde\varphi^2)\otimes\xi+\tilde\sigma(e_1)\eta\otimes e_2, \tag{4.22}$$

where $\ddot a$ and $\ddot b$ are smooth functions defined by $\ddot a=1+\frac{r}{2}$ and $\ddot b=-3-\frac{r}{2}$, respectively.

Proof. For 3-dimensional case, we have

$$\tilde lX=\tilde S(\xi,\xi)X-\tilde S(X,\xi)\xi+\tilde QX-\eta(X)\tilde Q\xi-\frac{r}{2}(X-\eta(X)\xi),$$

where $\tilde l$ denotes the Jacobi operator and X is a vector field. Using (2.5), the last equation implies

$$\tilde QX=-\tilde\varphi^2X+\tilde h^2X-\tilde\varphi(\tilde\nabla_\xi\tilde h)X-\tilde S(\xi,\xi)X+\tilde S(X,\xi)\xi+\eta(X)\tilde Q\xi+\frac{r}{2}(X-\eta(X)\xi). \tag{4.23}$$

Using $\tilde\varphi^2=I-\eta\otimes\xi$, we get $\tilde S(X,\xi)=\tilde S(\tilde\varphi^2X,\xi)+\eta(X)\tilde S(\xi,\xi)$. So (4.23) becomes

$$\tilde QX=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2X-\tilde\varphi(\tilde\nabla_\xi\tilde h)X-\tilde S(\xi,\xi)X+\tilde S(\tilde\varphi^2X,\xi)\xi+\eta(X)\tilde S(\xi,\xi)\xi+\eta(X)\tilde Q\xi+\frac{r}{2}\tilde\varphi^2X. \tag{4.24}$$

By the basis pseudo-orthonormal $\{e_1,e_2,\xi\}$ and (4.18), it follows that

$$\tilde Q\xi=\tilde\sigma(e_1)e_2+\tilde S(\xi,\xi)\xi. \tag{4.25}$$

Using (4.25) in (4.24), we have

$$\begin{aligned}\tilde QX &= \left(1+\frac{r}{2}\right)X+\left(-3-\frac{r}{2}\right)\eta(X)\xi\\ &\quad -\tilde\varphi(\tilde\nabla_\xi\tilde h)X+\tilde\sigma(\tilde\varphi^2X)\xi+\eta(X)\tilde\sigma(e_1)e_2,\end{aligned} \tag{4.26}$$

for arbitrary vector field X. Hence, proof comes from (4.26). □

A consequence of Lemma 4.10, we can give the components of the Ricci operator $\tilde Q$ by following,

$$\begin{aligned}\tilde Q\xi &= (\ddot a+\ddot b)\xi+\tilde\sigma(e_1)e_2,\\ \tilde Qe_1 &= \tilde\sigma(e_1)\xi+\ddot ae_1-2ae_2,\\ \tilde Qe_2 &= \ddot ae_2.\end{aligned} \tag{4.27}$$

The tensor $\tilde h$ has the canonical form (III). Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a 3-dimensional paracontact metric manifold and p is a point of M. Then there exists a local orthonormal $\tilde\varphi$-basis $\{\tilde e,\tilde\varphi\tilde e,\xi\}$ in a neighborhood of p where $-\tilde g(\tilde e,\tilde e)=\tilde g(\tilde\varphi\tilde e,\tilde\varphi\tilde e)=\tilde g(\xi,\xi)=1$. Now, let U_1 be the open subset of M where $\tilde h\neq0$ and let U_2 be the open subset of points $p\in M$ such that $\tilde h=0$ in a neighborhood of p. $U_1\cup U_2$ is an open subset of M. For every $p\in U_1$ there exists an open neighborhood of p such that

$\tilde{h}\tilde{e}=\tilde{\lambda}\tilde{\varphi}\tilde{e}, \tilde{h}\tilde{\varphi}\tilde{e}=-\tilde{\lambda}\tilde{e}$ and $\tilde{h}\xi=0$ where $\tilde{\lambda}$ is a non-vanishing smooth function. Since $tr\tilde{h}=0$, the matrix form of $\tilde{h}$ is given by

$$\tilde{h}=\begin{pmatrix} 0 & -\tilde{\lambda} & 0 \\ \tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.28}$$

with respect to local orthonormal basis $\{\tilde{e},\tilde{\varphi}\tilde{e},\xi\}$. In this case, we say that $\tilde{h}$ is of $\mathfrak{h}_3$ *type.*

Lemma 4.11. *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold with $\tilde{h}$ of $\mathfrak{h}_3$ type. Then for the covariant derivative on U_1 the following equations are valid*

$$\begin{aligned} i)\ \tilde{\nabla}_{\tilde{e}}\tilde{e} &= a_3\tilde{\varphi}\tilde{e}+\tilde{\lambda}\xi, \quad ii)\ \tilde{\nabla}_{\tilde{e}}\tilde{\varphi}\tilde{e}=a_3\tilde{e}+\xi, \quad iii)\ \tilde{\nabla}_{\tilde{e}}\xi=-\tilde{\varphi}\tilde{e}+\tilde{\lambda}\tilde{e}, \\ iv)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{e} &= b_3\tilde{\varphi}\tilde{e}-\xi, \quad v)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{e}=b_3\tilde{e}+\tilde{\lambda}\xi, \quad vi)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\xi=-\tilde{e}-\tilde{\lambda}\tilde{\varphi}\tilde{e}, \\ vii)\ \tilde{\nabla}_{\xi}\tilde{e} &= \tilde{b}_3\tilde{\varphi}\tilde{e}, \quad viii)\ \tilde{\nabla}_{\xi}\tilde{\varphi}\tilde{e}=\tilde{b}_3\tilde{e}, \\ ix)\ [\tilde{e},\xi] &= \tilde{\lambda}\tilde{e}-(1+\tilde{b}_3)\tilde{\varphi}\tilde{e}, \quad x)\ [\tilde{\varphi}\tilde{e},\xi]=-(1+\tilde{b}_3)\tilde{e}-\tilde{\lambda}\tilde{\varphi}\tilde{e}, \quad xi)\ [\tilde{e},\tilde{\varphi}\tilde{e}]=a_3\tilde{e}-b_3\tilde{\varphi}\tilde{e}+2\xi. \end{aligned} \tag{4.29}$$

where a_3, b_3 and $\tilde{b}_3$ are defined by

$$a_3=-\frac{1}{2\tilde{\lambda}}\left[\sigma(\tilde{\varphi}\tilde{e})+(\tilde{\varphi}\tilde{e})(\tilde{\lambda})\right], \quad \tilde{\sigma}(\tilde{e})=S(\xi,\tilde{e}),$$

$$b_3=\frac{1}{2\tilde{\lambda}}\left[\sigma(\tilde{e})-\tilde{e}(\tilde{\lambda})\right], \quad \tilde{\sigma}(\tilde{\varphi}\tilde{e})=S(\xi,\tilde{\varphi}\tilde{e}),$$

$$\tilde{b}_3=\tilde{g}(\tilde{\nabla}_\xi\tilde{e},\tilde{\varphi}\tilde{e})$$

respectively.

Proof. Replacing X by $\tilde{e}$ and Y by $\tilde{\varphi}\tilde{e}$ in equation $\tilde{\nabla}\xi=-\tilde{\varphi}+\tilde{\varphi}\tilde{h}$, we have $iii), vi)$.

For the proof of $viii)$ we have

$$\begin{aligned} \tilde{\nabla}_\xi\tilde{\varphi}\tilde{e} &= -\tilde{g}(\tilde{\nabla}_\xi\tilde{\varphi}\tilde{e},\tilde{e})\tilde{e}+\tilde{g}(\tilde{\nabla}_\xi\tilde{\varphi}\tilde{e},\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e}+\tilde{g}(\tilde{\nabla}_\xi\tilde{\varphi}\tilde{e},\xi)\xi \\ &= \tilde{g}(\tilde{\varphi}\tilde{e},\tilde{\nabla}_\xi\tilde{e})\tilde{e}, \end{aligned}$$

where $\tilde{b}_3=\tilde{g}(\tilde{\nabla}_\xi\tilde{e},\tilde{\varphi}\tilde{e})$. So we obtain $\tilde{\nabla}_\xi\tilde{\varphi}\tilde{e}=\tilde{b}_3\tilde{e}$. The proofs of other covariant derivative equalities are similar to $viii)$.

Putting $X=\tilde{e}$, $Y=\tilde{\varphi}\tilde{e}$, $Z=\xi$ in the equation (4.1), we have

$$\tilde{R}(\tilde{e},\tilde{\varphi}\tilde{e})\xi=-\tilde{g}(\tilde{Q}\tilde{e},\xi)\tilde{\varphi}\tilde{e}+\tilde{g}(\tilde{Q}\tilde{\varphi}\tilde{e},\xi)\tilde{e}.$$

Since $\tilde{\sigma}(X)$ is defined as $\tilde{g}(\tilde{Q}\xi,X)$, we have

$$\tilde{R}(\tilde{e},\tilde{\varphi}\tilde{e})\xi=-\tilde{\sigma}(\tilde{e})\tilde{\varphi}\tilde{e}+\tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{e}. \tag{4.30}$$

On the other hand, by using (2.4), we have

$$\begin{aligned} \tilde{R}(\tilde{e},\tilde{\varphi}\tilde{e})\xi &= (\tilde{\nabla}_{\tilde{e}}\tilde{\varphi}\tilde{h})\tilde{\varphi}\tilde{e}-(\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{h})\tilde{e} \\ &= (-2a_3\tilde{\lambda}-(\tilde{\varphi}\tilde{e})(\tilde{\lambda}))\tilde{e}+(-2b_3\tilde{\lambda}-\tilde{e}(\tilde{\lambda}))\tilde{\varphi}\tilde{e}. \end{aligned} \tag{4.31}$$

Comparing (4.31) with (4.30), we get

$$\tilde{\sigma}(\tilde{e})=\tilde{e}(\tilde{\lambda})+2b_3\tilde{\lambda}, \quad \tilde{\sigma}(\tilde{\varphi}\tilde{e})=-(\tilde{\varphi}\tilde{e})(\tilde{\lambda})-2a_3\tilde{\lambda}.$$

Hence, the functions a_3 and b_3 are obtained from the last equation. □

Next, we derive a useful formula for $\tilde{\nabla}_\xi\tilde{h}$.

Proposition 4.12. *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a 3-dimensional paracontact metric manifold with $\tilde h$ of $\mathfrak{h}_3$ type. So, on U_1 we have*

$$\tilde\nabla_\xi\tilde h=-2\tilde b_3\tilde h\tilde\varphi+\xi(\tilde\lambda)s, \tag{4.32}$$

where s is the $(1,1)$-type tensor defined by $s\xi=0$, $s\tilde e=\tilde\varphi\tilde e$, $s\tilde\varphi\tilde e=-\tilde e$.

Proof. Using (4.29), we get

$$\begin{aligned}
(\tilde\nabla_\xi\tilde h)\xi &= 0=(-2\tilde b_3\tilde h\tilde\varphi+\xi(\tilde\lambda)s)\xi,\\
(\tilde\nabla_\xi\tilde h)\tilde e &= 2\tilde b_3\tilde\lambda\tilde e+\xi(\tilde\lambda)\tilde\varphi\tilde e=(-2\tilde b_3\tilde h\tilde\varphi+\xi(\tilde\lambda)s)\tilde e,\\
(\tilde\nabla_\xi\tilde h)\tilde\varphi\tilde e &= -2\tilde b_3\tilde\lambda\tilde\varphi\tilde e-\xi(\tilde\lambda)\tilde e=(-2\tilde b_3\tilde h\tilde\varphi+\xi(\tilde\lambda)s)\tilde\varphi\tilde e.
\end{aligned}$$

The last equations complete the proof of (4.32). □

Proposition 4.13. *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a 3-dimensional paracontact metric manifold with $\tilde h$ of $\mathfrak{h}_3$ type. Then the following equation holds on M.*

$$\tilde h^2-\tilde\varphi^2=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2. \tag{4.33}$$

Proof. Using (2.6), we have $\tilde S(\xi,\xi)=2(1+\tilde\lambda^2)$. After calculating $\tilde h^2-\tilde\varphi^2$ with respect to the basis components, we get

$$\tilde h^2\xi-\tilde\varphi^2\xi=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2\xi=0,\quad \tilde h^2\tilde e-\tilde\varphi^2\tilde e=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2\tilde e,\quad \tilde h^2\tilde\varphi\tilde e-\tilde\varphi^3\tilde e=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2\tilde\varphi\tilde e. \tag{4.34}$$

(4.34) completes the proof of (4.33). □

Lemma 4.14. *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a 3-dimensional paracontact metric manifold with $\tilde h$ of $\mathfrak{h}_3$ type. Then the Ricci operator $\tilde Q$ is given by*

$$\tilde Q=\bar a\ I+\bar b\eta\otimes\xi-\tilde\varphi(\tilde\nabla_\xi\tilde h)+\tilde\sigma(\tilde\varphi^2)\otimes\xi-\tilde\sigma(\tilde e)\eta\otimes\tilde e+\tilde\sigma(\tilde\varphi\tilde e)\eta\otimes\tilde\varphi\tilde e, \tag{4.35}$$

where $\bar a$ and $\bar b$ are smooth functions defined by $\bar a=1+\tilde\lambda^2+\frac{r}{2}$ and $\bar b=-3(\tilde\lambda^2+1)-\frac{r}{2}$, respectively. Moreover the components of the Ricci operator $\tilde Q$ are given by

$$\begin{aligned}
\tilde Q\xi &= (\bar a+\bar b)\xi-\tilde\sigma(\tilde e)\tilde e+\tilde\sigma(\tilde\varphi\tilde e)\tilde\varphi\tilde e,\\
\tilde Q\tilde e &= \tilde\sigma(\tilde e)\xi+(\bar a+\xi(\tilde\lambda))\tilde e-2\tilde b_3\tilde\lambda\tilde\varphi\tilde e,\\
\tilde Q\tilde\varphi\tilde e &= \tilde\sigma(\tilde\varphi\tilde e)\xi+2\tilde b_3\tilde\lambda\tilde e+(\bar a+\xi(\tilde\lambda))\tilde\varphi\tilde e.
\end{aligned} \tag{4.36}$$

Proof. By (4.1), we have

$$\tilde R(X,\xi)\xi=\tilde S(\xi,\xi)X-\tilde S(X,\xi)\xi+\tilde QX-\eta(X)\tilde Q\xi-\frac{r}{2}(X-\eta(X)\xi),$$

for any vector field X. Using (2.5), the last equation implies

$$\tilde QX=-\tilde\varphi^2X+\tilde h^2X-\tilde\varphi(\tilde\nabla_\xi\tilde h)X-\tilde S(\xi,\xi)X+\tilde S(X,\xi)\xi+\eta(X)\tilde Q\xi+\frac{r}{2}(X-\eta(X)\xi). \tag{4.37}$$

By writing $\tilde S(X,\xi)=\tilde S(\tilde\varphi^2X,\xi)+\eta(X)\tilde S(\xi,\xi)$ in (4.37), we obtain

$$\tilde QX=\frac{\tilde S(\xi,\xi)}{2}\tilde\varphi^2X-\tilde\varphi(\tilde\nabla_\xi\tilde h)X-\tilde S(\xi,\xi)X+\tilde S(\tilde\varphi^2X,\xi)\xi+\eta(X)\tilde S(\xi,\xi)\xi+\eta(X)\tilde Q\xi+\frac{r}{2}\tilde\varphi^2X. \tag{4.38}$$

We know that the Ricci tensor $\tilde S$ with respect to the orthonormal basis $\{\tilde e,\tilde\varphi\tilde e,\xi\}$ is given by

$$\tilde Q\xi=-\tilde\sigma(\tilde e)\tilde e+\tilde\sigma(\tilde\varphi\tilde e)\tilde\varphi\tilde e+\tilde S(\xi,\xi)\xi. \tag{4.39}$$

Using (4.39) in (4.38), we have

$$\tilde{Q}X = \Big(1+\tilde{\lambda}^2+\frac{r}{2}\Big)X+\Big(-3(\tilde{\lambda}^2+1)-\frac{r}{2}\Big)\eta(X)\xi \\ -\tilde{\varphi}(\tilde{\nabla}_{\xi}\tilde{h})X+\tilde{\sigma}(\tilde{\varphi}^2X)\xi-\eta(X)\tilde{\sigma}(\tilde{e})\tilde{e}+\eta(X)\tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e}, \tag{4.40}$$

for arbitrary vector field X. This ends the proof. □

The tensor $\tilde{h}$ has the canonical form (IV). Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold and p is a point of M. Then there exists a local pseudo-orthonormal basis $\{e_1,e_2,e_3\}$ in a neighborhood of p where $\tilde{g}(e_1,e_1)=\tilde{g}(e_2,e_2)=\tilde{g}(e_1,e_3)=\tilde{g}(e_2,e_3)=0$ and $\tilde{g}(e_1,e_2)=\tilde{g}(e_3,e_3)=1$. Since the tensor $\tilde{h}$ has canonical form (IV) (with respect to a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$) then $\tilde{h}e_1=\tilde{\lambda}e_1+e_3$, $\tilde{h}e_2=\tilde{\lambda}e_2$ and $\tilde{h}e_3=e_2+\tilde{\lambda}e_3$. Since $0=tr\tilde{h}=\tilde{g}(\tilde{h}e_1,e_2)+\tilde{g}(\tilde{h}e_2,e_1)+\tilde{g}(\tilde{h}e_3,e_3)=3\tilde{\lambda}$, then $\tilde{\lambda}=0$. We write $\xi=\tilde{g}(\xi,e_2)e_1+\tilde{g}(\xi,e_1)e_2+\tilde{g}(\xi,e_3)e_3$ respect to the pseudo-orthonormal basis $\{e_1,e_2,e_3\}$. Since $\tilde{h}\xi=0$, we have $0=\tilde{g}(\xi,e_2)e_3+\tilde{g}(\xi,e_3)e_2$. Hence we get $\xi=\tilde{g}(\xi,e_1)e_2$ which leads to a contradiction with $\tilde{g}(\xi,\xi)=1$. Thus, this case does not occur.

The proof of Theorem 1.1:

According to a suitable (pseudo)-orthonormal basis, $\tilde{h}$ can be put into one of the three forms. So we will need three cases.

Case 1: Firstly, we suppose that $\tilde{h}$ is $\mathfrak{h}_1$ type.

From (1.3) we have

$$\tilde{\nabla}^*\tilde{\nabla}\xi = -\tilde{\nabla}_{\tilde{e}}\tilde{\nabla}_{\tilde{e}}\xi+\tilde{\nabla}_{\tilde{\nabla}_{\tilde{e}}\tilde{e}}\xi+\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\xi-\tilde{\nabla}_{\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{e}}\xi \\ = -\tilde{\sigma}(\tilde{e})\tilde{e}+\tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e}+2(\tilde{\lambda}^2+1)\xi.$$

By using (1.4) and (4.8) we obtain $\tilde{\sigma}(\tilde{e})=\tilde{\sigma}(\tilde{\varphi}\tilde{e})=0$ and $\tilde{Q}\xi=2(\tilde{\lambda}^2-1)\xi$.

Case 2: Secondly, we assume that $\tilde{h}$ is $\mathfrak{h}_2$ type.

We construct an orthonormal basis $\{\tilde{e},\tilde{\varphi}\tilde{e},\xi\}$ from the pseudo-orthonormal basis $\{e_1,e_2,\xi\}$ such that

$$\tilde{e}=\frac{e_1-e_2}{\sqrt{2}},\quad \tilde{\varphi}\tilde{e}=\frac{e_1+e_2}{\sqrt{2}},\ \tilde{g}(\tilde{e},\tilde{e})=-1 \text{ and } \tilde{g}(\tilde{\varphi}\tilde{e},\tilde{\varphi}\tilde{e})=1. \tag{4.41}$$

Then $\tilde{h}$ with respect to this new basis takes the form,

$$\tilde{h}\tilde{e}=\tilde{h}\tilde{\varphi}\tilde{e}=\frac{1}{2}(-\tilde{e}+\tilde{\varphi}\tilde{e}). \tag{4.42}$$

By Lemma 4.7 we have

$$\begin{aligned}\tilde{\nabla}_{\tilde{e}}\tilde{e} &= \frac{1}{\sqrt{2}}(-b_2-\frac{1}{2}\tilde{\sigma}(e_1))\tilde{\varphi}\tilde{e}+\frac{1}{2}\xi,\ \tilde{\nabla}_{\tilde{e}}\tilde{\varphi}\tilde{e}=-\frac{1}{\sqrt{2}}(b_2+\frac{1}{2}\tilde{\sigma}(e_1))\tilde{e}+\frac{3}{2}\xi,\\ \tilde{\nabla}_{\tilde{e}}\xi &= \frac{\tilde{e}-3\tilde{\varphi}\tilde{e}}{2},\quad \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{e}=-\frac{1}{\sqrt{2}}(b_2-\frac{1}{2}\tilde{\sigma}(e_1))\tilde{\varphi}\tilde{e}-\frac{1}{2}\xi,\\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{e} &= -\frac{1}{\sqrt{2}}(b_2-\frac{1}{2}\tilde{\sigma}(e_1))\tilde{e}+\frac{1}{2}\xi,\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\xi=\frac{-\tilde{e}-\tilde{\varphi}\tilde{e}}{2}\quad .\end{aligned} \tag{4.43}$$

Using (1.3) and (4.43) we get

$$\tilde{\nabla}^*\tilde{\nabla}\xi = -\tilde{\nabla}_{\tilde{e}}\tilde{\nabla}_{\tilde{e}}\xi+\tilde{\nabla}_{\tilde{\nabla}_{\tilde{e}}\tilde{e}}\xi+\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\xi-\tilde{\nabla}_{\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{e}}\xi \\ = \frac{\tilde{\sigma}(e_1)}{\sqrt{2}}(\tilde{e}-\tilde{\varphi}\tilde{e})+2\xi. \tag{4.44}$$

From (1.4) and (4.44), we obtain $\tilde{\sigma}(e_1)=0$. By help of (4.27), ξ is an eigenvector of the Ricci operator.

Case 3: Finally, let $\tilde{h}$ be $\mathfrak{h}_3$ type.

Again using (1.3) we obtain

$$\begin{aligned}\tilde{\nabla}^{*}\tilde{\nabla}\xi &= -\tilde{\nabla}_{\tilde{e}}\tilde{\nabla}_{\tilde{e}}\xi + \tilde{\nabla}_{\tilde{\nabla}_{\tilde{e}}\tilde{e}}\xi + \tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\xi - \tilde{\nabla}_{\tilde{\nabla}_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{e}}\xi \\ &= -\tilde{\sigma}(\tilde{e})\tilde{e} + \tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e} + 2(1-\tilde{\lambda}^{2})\xi.\end{aligned}$$

From (1.4), we obtain $\tilde{\sigma}(\tilde{e}) = \tilde{\sigma}(\tilde{\varphi}\tilde{e}) = 0$. By (4.36) we have $\tilde{Q}\xi = -2(1+\tilde{\lambda}^{2})\xi$.

This completes the proof.

The proof of Theorem 1.2:

We give proof of the theorem for three cases respect to chosen (pseudo)-orthonormal basis.

Case 1: We assume that $\tilde{h}$ is $\mathfrak{h}_1$ type.

Since ξ is a harmonic vector field, ξ is an eigenvector of $\tilde{Q}$. Hence we obtain that $\tilde{\sigma} = 0$. Putting $s = \frac{1}{\tilde{\lambda}}\tilde{h}$ in (4.7) we have

$$\tilde{Q} = a_1 I + b_1\eta\otimes\xi - 2b\tilde{h} - \frac{\xi(\tilde{\lambda})}{\tilde{\lambda}}\tilde{\varphi}\tilde{h}. \tag{4.45}$$

Setting $Z = \xi$ in (4.1) and using (4.45), we obtain

$$\tilde{R}(X,Y)\xi = (\tilde{\lambda}^{2}-1)(\eta(Y)X - \eta(X)Y) - 2b(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) - \frac{\xi(\tilde{\lambda})}{\tilde{\lambda}}(\eta(Y)\tilde{\varphi}\tilde{h}X - \eta(X)\tilde{\varphi}\tilde{h}Y),$$

where the functions $\tilde{\kappa}$, $\tilde{\mu}$ and $\tilde{\nu}$ defined by $\tilde{\kappa} = \frac{\tilde{S}(\xi,\xi)}{2}$, $\tilde{\mu} = -2b$, $\tilde{\nu} = -\frac{\xi(\tilde{\lambda})}{\tilde{\lambda}}$, respectively. So, it is obvious that for this type $\tilde{\kappa} > -1$. Moreover, using (4.45), we have $\tilde{Q}\tilde{\varphi} - \tilde{\varphi}\tilde{Q} = 2\tilde{\mu}\tilde{h}\tilde{\varphi} - 2\tilde{\nu}\tilde{h}$.

Case 2: Secondly, let $\tilde{h}$ be $\mathfrak{h}_2$ type.

Putting $\tilde{\sigma} = 0$ in (4.22) we get

$$\tilde{Q} = \ddot{a}I + \ddot{b}\eta\otimes\xi - 2a_2\tilde{h}, \tag{4.46}$$

which yields

$$\tilde{Q}\xi = \tilde{S}(\xi,\xi)\xi, \tag{4.47}$$

for any vector fields on M. Putting ξ instead of Z in (4.1) we obtain

$$\begin{aligned}\tilde{R}(X,Y)\xi &= -\tilde{S}(X,\xi) + \tilde{S}(Y,\xi) - \eta(X)\tilde{Q}Y \\ &\quad +\eta(Y)\tilde{Q}X + \frac{r}{2}(\eta(X)Y - \eta(Y)X),\end{aligned} \tag{4.48}$$

for any vector field X. Using (4.46) and (4.47) in (4.48), we obtain

$$\tilde{R}(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) - 2a_2(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y),$$

where the functions $\tilde{\kappa}$ and $\tilde{\mu}$ defined by $\tilde{\kappa} = \frac{\tilde{S}(\xi,\xi)}{2}$, $\tilde{\mu} = -2a_2$, respectively. So, it is obvious that for this type $\tilde{\kappa} = -1$. Furthermore, by (4.46), we have $\tilde{Q}\tilde{\varphi} - \tilde{\varphi}\tilde{Q} = 2\tilde{\mu}\tilde{h}\tilde{\varphi}$.

Case 3: Finally, we suppose that $\tilde{h}$ is $\mathfrak{h}_3$ type.

Since M is a H-paracontact metric manifold we have $\tilde{\sigma} = 0$. Putting $s = \frac{1}{\tilde{\lambda}}\tilde{h}$ in (4.35) we get

$$\tilde{Q} = \bar{a}I + \bar{b}\eta\otimes\xi - 2\tilde{b}_3\tilde{h} - \left(\frac{\xi(\tilde{\lambda})}{\tilde{\lambda}}\right)\tilde{\varphi}\tilde{h}, \tag{4.49}$$

which yields

$$\tilde{Q}\xi = \tilde{S}(\xi,\xi)\xi, \tag{4.50}$$

for any vector fields on M. Setting $\xi = Z$ in (4.1) we find

$$\begin{aligned}\tilde{R}(X,Y)\xi &= -\tilde{S}(X,\xi)+\tilde{S}(Y,\xi)-\eta(X)\tilde{Q}Y \\ &\quad +\eta(Y)\tilde{Q}X+\frac{r}{2}(\eta(X)Y-\eta(Y)X),\end{aligned} \tag{4.51}$$

for any vector field X. Using (4.49) and (4.50) in (4.51), we get

$$\tilde{R}(X,Y)\xi = (-1-\tilde{\lambda}^2)(\eta(Y)X-\eta(X)Y)-2\tilde{b}_3(\eta(Y)\tilde{h}X-\eta(X)\tilde{h}Y)-\frac{\xi(\tilde{\lambda})}{\tilde{\lambda}}(\eta(Y)\tilde{\varphi}\tilde{h}X-\eta(X)\tilde{\varphi}\tilde{h}Y),$$

where the functions $\tilde{\kappa}$, $\tilde{\mu}$ and $\tilde{\nu}$ defined by $\tilde{\kappa} = \frac{\tilde{S}(\xi,\xi)}{2}$, $\tilde{\mu} = -2\tilde{b}_3$, $\tilde{\nu} = -\frac{\xi(\tilde{\lambda})}{\tilde{\lambda}}$, respectively. So, it is obvious that for this type $\tilde{\kappa} < -1$. By help of (4.49), we get $\tilde{Q}\tilde{\varphi}-\tilde{\varphi}\tilde{Q} = -2\tilde{\mu}\tilde{h}\tilde{\varphi}-2\tilde{\nu}\tilde{h}$.

Conversely, let M is a paracontact metric ($\tilde{\kappa},\tilde{\mu},\tilde{\nu}$)-manifold. Using Teorem 1.1 and (3.3), we conclude that ξ is harmonic vector field.

This completes the proof.

Concluding Lemma: *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold. Then a canonical form of $\tilde{h}$ stays constant in an open neighborhood of any point for $\tilde{h}$.*

Proof: Let U_1 be open subset of M where $\tilde{h} \neq 0$ and $p,q \in U_1$ with $p \neq q$.

Case 1. Now we assume that $\tilde{h}_p$ has canonical form (I) at $T_pM_1^3$. In this case, there exists an orthonormal $\tilde{\varphi}$-basis $\{\tilde{e},\tilde{\varphi}\tilde{e},\xi\}$ such that

$$\tilde{h}_p\tilde{e} = \tilde{\lambda}(p)\tilde{e},\ \ \tilde{h}_p(\tilde{\varphi}\tilde{e}) = -\ \tilde{\lambda}(p)\tilde{\varphi}\tilde{e},\quad \tilde{h}_p\xi = 0, \tag{4.52}$$

with $-\tilde{g}(\tilde{e},\tilde{e}) = \tilde{g}(\tilde{\varphi}\tilde{e},\tilde{\varphi}\tilde{e}) = \tilde{g}(\xi,\xi) = 1$.

In $T_qM_1^3$ we suppose that $\tilde{h}_q$ has canonical form (II). By Lemma 4.5 we can construct a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$ in a neighborhood of q such that $\tilde{h}_qe_1 = e_2, \tilde{h}_qe_2 = 0, \tilde{h}_qe_3 = 0$ and $\tilde{\varphi}e_1 = e_1$, $\tilde{\varphi}e_2 = -e_2, \tilde{\varphi}e_3 = 0$ and also $\xi = e_3$ with $\tilde{g}(e_1,e_1) = \tilde{g}(e_2,e_2) = \tilde{g}(e_1,e_3) = \tilde{g}(e_2,e_3) = 0$ and $\tilde{g}(e_1,e_2) = \tilde{g}(e_3,e_3) = 1$. Putting

$$\tilde{E} = \frac{e_1-e_2}{\sqrt{2}},\qquad \tilde{\varphi}\tilde{E} = \frac{e_1+e_2}{\sqrt{2}} \tag{4.53}$$

we get an orthonormal basis such that $-\tilde{g}(\tilde{E},\tilde{E}) = \tilde{g}(\tilde{\varphi}\tilde{E},\tilde{\varphi}\tilde{E}) = \tilde{g}(\xi,\xi) = 1$ and also

$$\tilde{h}_q\tilde{E} = \tilde{h}_q\tilde{\varphi}\tilde{E} = \frac{1}{2}(-\tilde{E}+\tilde{\varphi}\tilde{E}). \tag{4.54}$$

So tangent spaces $T_pM = span\{\tilde{e},\tilde{\varphi}\tilde{e},\xi\}$ and $T_qM = span\{\tilde{E},\tilde{\varphi}\tilde{E},\xi)$ have same dimension and index. Thus there exist a linear isometry F_* from T_pM to T_qM such that $F(p) = q$ and

$$F_*(\tilde{e}) = \tilde{E},\quad F_*(\tilde{\varphi}\tilde{e}) = \tilde{\varphi}\tilde{E},\quad F_*(\xi) = \xi. \tag{4.55}$$

By (4.52) and (4.55) we have

$$F_*(\tilde{h}_p\tilde{e}) = \tilde{\lambda}(q)\tilde{E},\ F_*(\tilde{h}_p\tilde{\varphi}\tilde{e})\ = -\tilde{\lambda}(q)\tilde{\varphi}\tilde{E}. \tag{4.56}$$

Using (4.54) and (4.56) we obtain

$$0 = tr\tilde{h}_q^2 = 2\tilde{\lambda}^2(q),$$

which implies that $\tilde{\lambda}(q) = 0$. Hence we contradict the fact that $q \in U_1$.

Case 2. Again we assume that $\tilde{h}_p$ has canonical form (I) at $T_pM_1^3$. Suppose to contrary that $\tilde{h}_q$ has canonical form (III) in T_qM. One can construct a local orthonormal $\tilde{\varphi}$-basis $\{\tilde{f}_1,\tilde{\varphi}\tilde{f}_1,\xi\}$ in a neighborhood of q such that $-\tilde{g}(\tilde{f}_1,\tilde{f}_1) = \tilde{g}(\tilde{\varphi}\tilde{f}_1,\tilde{\varphi}\tilde{f}_1) = \tilde{g}(\xi,\xi) = 1, \tilde{h}_q(\tilde{f}_1) = \tilde{\lambda}_1(q)\tilde{\varphi}\tilde{f}_1, \tilde{h}_q(\tilde{\varphi}\tilde{f}_1) = -\tilde{\lambda}_1(q)\tilde{f}_1$. Since tangent spaces $T_pM_1^3 = span\{\tilde{e},\tilde{\varphi}\tilde{e},\xi\}$ and $T_qM_1^3 = span\{\tilde{f}_1,\tilde{\varphi}\tilde{f}_1,\xi)$ have same dimension and index, we can construct a linear isometry T_* from $T_pM_1^3$ to $T_qM_1^3$ such that $T(p)=q$ and

$$T_*(\tilde{e}) = \tilde{f}_1, \quad T_*(\tilde{\varphi}\tilde{e}) = \tilde{\varphi}\tilde{f}_1, \quad T_*(\xi) = \xi. \tag{4.57}$$

So we get

$$T_*(\tilde{h}_p\tilde{e}) = \tilde{\lambda}(q)\tilde{f}_1, T_*(\tilde{h}_p\tilde{\varphi}\tilde{e}) \ = -\tilde{\lambda}(q)\tilde{\varphi}\tilde{f}_1. \tag{4.58}$$

From (4.57) and (4.58) we find

$$-2\tilde{\lambda}_1^2(q) = tr\tilde{h}_q^2 = 2\tilde{\lambda}^2(q)$$

and this last equation gives $\tilde{\lambda}_1(q) = \tilde{\lambda}(q) = 0$ which contradicts with $q \in U_1$.

Case 3. Let us consider that $\tilde{h}_p$ has canonical form (III) at $T_pM_1^3$. In this case, there exist an orthonormal $\tilde{\varphi}$-basis $\{\tilde{e},\tilde{\varphi}\tilde{e},\xi\}$ such that

$$\tilde{h}_p\tilde{e} = \tilde{\lambda}(p)\tilde{\varphi}\tilde{e}, \ \ \tilde{h}_p(\tilde{\varphi}\tilde{e}) = -\tilde{\lambda}(p)\tilde{e}, \quad \tilde{h}_p\xi = 0$$

with $-\tilde{g}(\tilde{e},\tilde{e}) = \tilde{g}(\tilde{\varphi}\tilde{e},\tilde{\varphi}\tilde{e}) = \tilde{g}(\xi,\xi) = 1$. In $T_qM_1^3$ we suppose that $\tilde{h}_q$ has canonical form (II). Similar arguments as in Case 1, we obtain

$$0 = tr\tilde{h}_q^2 = 2\tilde{\lambda}^2(q)$$

which leads to a contradiction of chosen q. This completes proof of the concluding lemma.

Now we will give some examples of 3-dimensional $(\tilde{\kappa},\,\tilde{\mu},\tilde{\nu})$-paracontact metric manifolds according to the cases $\tilde{\kappa} > -1$, $\tilde{\kappa} = -1$ and $\tilde{\kappa} < -1$.

Example 4.15. *We consider the* 3*-dimensional manifold*

$$M = \{(x,y,z) \in R^3 \mid 2y+z \neq 0, \ \ z \neq 0\}$$

and the vector fields

$$e_1 = \ \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = (2y+z)\frac{\partial}{\partial x} - (2zx - \frac{1}{2z}y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1*-form* $\eta = dx - (2y+z)dz$ *defines a contact structure on* M *with characteristic vector field* $\xi = \frac{\partial}{\partial x}$. *We define the structure tensor* $\tilde{\varphi}\tilde{e}_1 = 0$, $\tilde{\varphi}\tilde{e}_2 = \tilde{e}_3$ *and* $\tilde{\varphi}\tilde{e}_3 = \tilde{e}_2$. *Let* $\tilde{g}$ *be Lorentzian metric defined by* $\tilde{g}(e_1,e_1) = -\tilde{g}(e_2,e_2) = \tilde{g}(e_3,e_3) = 1$ *and* $\tilde{g}(e_1,e_2) = \tilde{g}(e_1,e_3) = \tilde{g}(e_2,e_3) = 0$. *Then* $(\tilde{\varphi},\xi,\eta,\tilde{g})$ *is a paracontact metric structure on* M. *Using Lemma 4.1, we conclude that* M *is a generalized* $(\tilde{\kappa},\tilde{\mu})$ *paracontact metric manifold with* $\tilde{\kappa} = -1+z^2, \tilde{\mu} = 2(1-z)$.

Example 4.16. *Consider the* 3*-dimensional manifold*

$$M = \left\{(x,y,z) \in \mathbb{R}^3 \mid 2y - z \neq 0, \right\},$$

where (x,y,z) *are the cartesian coordinates in* $\mathbb{R}^3$. *We define three vector fields on* M *as*

$$e_1 = (-2y+z)\frac{\partial}{\partial x} + (x-2y-z)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial x}.$$

The pseudo-Riemannian metric $\tilde{g}$, *and the* $(1,1)$*-tensor field* $\tilde{\varphi}$ *given by*

$$\tilde{g} = \begin{pmatrix} 1 & 0 & (2y-z)/2 \\ 0 & 0 & \frac{1}{2} \\ (2y-z)/2 & \frac{1}{2} & (-2y+z)^2 - 2(x-2y-z) \end{pmatrix}, \quad \tilde{\varphi} \ = \begin{pmatrix} 0 & 0 & -2y+z \\ 0 & -1 & x-2y-z \\ 0 & 0 & 1 \end{pmatrix}.$$

So we easily obtain $\tilde{g}(e_1,e_1) = \tilde{g}(e_2,e_2) = \tilde{g}(e_1,e_3) = \tilde{g}(e_2,e_3) = 0$ *and* $\tilde{g}(e_1,e_2) = \tilde{g}(e_3,e_3) = 1$. *Moreover we have* $\eta = dx + (2y-z)dz$ *and* $\tilde{h} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *with respect to the basis* $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. *By direct calculations we get*

$$\tilde{R}(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) + 2(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y).$$

Finally we deduce that M *is a* $(-1,2,0)$*-paracontact metric manifold.*

Remark 4.17. To our knowledge, the above example is the first numerical example satisfying $\tilde{\kappa} = -1$ and $\tilde{h} \neq 0$ in $\mathbb{R}^3$.

Example 4.18. *In* [20] *Koufogiorgos et al. construct following example.*

Consider 3*-dimensional manifold*

$$M = \left\{(x,y,z) \in \mathbb{R}^3 \mid 2x + e^{y+z} > 0, \quad y \neq z\right\}$$

and the vector fields $e_1 = \frac{\partial}{\partial x}$,

$$\begin{aligned} e_2 &= \left(-\left(\frac{y^2+z^2}{2}\right)(2x+e^{y+z})^{\frac{1}{2}}\right)\frac{\partial}{\partial x} + \left(\frac{z(2x+e^{y+z})^{\frac{1}{2}}}{y-z} + \frac{(2x+e^{y+z})^{-\frac{1}{2}}}{y-z}\right)\frac{\partial}{\partial y} \\ &\quad + \left(\frac{y(2x+e^{y+z})^{\frac{1}{2}}}{z-y} + \frac{(2x+e^{y+z})^{-\frac{1}{2}}}{z-y}\right)\frac{\partial}{\partial z}, \\ e_3 &= \left(\left(\frac{y^2+z^2}{2}\right)(2x+e^{y+z})^{\frac{1}{2}}\right)\frac{\partial}{\partial x} + \left(\frac{z(2x+e^{y+z})^{\frac{1}{2}}}{z-y} + \frac{(2x+e^{y+z})^{-\frac{1}{2}}}{y-z}\right)\frac{\partial}{\partial y} \\ &\quad + \left(\frac{y(2x+e^{y+z})^{\frac{1}{2}}}{y-z} + \frac{(2x+e^{y+z})^{-\frac{1}{2}}}{z-y}\right)\frac{\partial}{\partial z}. \end{aligned}$$

Let η *be the* 1*-form dual to* e_1. *The contact Riemannian structure is defined as follows*

$$\xi = e_1, \varphi e_1 = 0, \varphi e_2 = e_3 \text{ and } \varphi e_3 = -e_2,$$
$$g(e_i,e_j) = \delta_{ij} \quad \text{for any } i,j \in \{1,2,3\}.$$

Thus it can be deduced that (M,φ,ξ,η,g) *is a* (κ,μ,ν)*-contact metric manifold with* $\kappa = 1 - \frac{1}{(2x+e^{y+z})^2}$, $\mu = 2$ *and* $\nu = \frac{-2}{(2x+e^{y+z})}$.

Next,using (5.11) we can construct paracontact structure as follows

$$\tilde{e}_1 = \xi, \tilde{e}_2 = \frac{1}{\sqrt{2}}(e_2 - e_3), \tilde{e}_3 = \frac{1}{\sqrt{2}}(e_2 + e_3) \text{ such that } \tilde{\varphi}\tilde{e}_1 = 0, \tilde{\varphi}\tilde{e}_2 = \tilde{e}_3 \text{ and } \tilde{\varphi}\tilde{e}_3 = \tilde{e}_2$$

$$\begin{aligned} \tilde{g}(\tilde{e}_1,\tilde{e}_1) &= 1, \quad \tilde{g}(\tilde{e}_2,\tilde{e}_2) = -1, \quad \tilde{g}(\tilde{e}_3,\tilde{e}_3) = 1 \text{ and} \\ \tilde{g}(\tilde{e}_1,\tilde{e}_2) &= \tilde{g}(\tilde{e}_1,\tilde{e}_3) = \tilde{g}(\tilde{e}_2,\tilde{e}_3) = 0. \end{aligned}$$

Moreover, the matrix form of $\tilde{h}$ *is given by*

$$\tilde{h} = \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\lambda = \sqrt{1-\kappa}$. *After calculations, we finally deduce that* $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ *is a* $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$*-paracontact metric manifold* $\tilde{\kappa} = \kappa - 2$, $\tilde{\mu} = 2$ *and* $\tilde{\nu} = -\nu$.

Remark 4.19. In the last example ν is a non-constant smooth function.

Remark 4.20. Choosing ν is a constant function, we can construct a family of $(\tilde{\kappa} < -1, \tilde{\mu} = 2, \tilde{\nu} = -\nu)$-paracontact metric manifolds.

We will finish this section by the following theorem.

Theorem 4.21. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold . If the characteristic vector field $\xi : (M, \tilde{g}) \to (T_1M, \tilde{g}^s)$ is harmonic map then paracontact $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold is paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold, i.e. $\tilde{\nu} = 0$.*

Proof. Using the definition of paracontact $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold and properties of curvature tensor one has

$$\tilde{R}(\xi, W)X = \tilde{\kappa}(\tilde{g}(X, W)\xi - \eta(X)W) + \tilde{\mu}(\tilde{g}(\tilde{h}X, W)\xi - \eta(X)\tilde{h}W) + \tilde{\nu}(\tilde{g}(\tilde{\varphi}\tilde{h}X, W)\xi - \eta(X)\tilde{\varphi}\tilde{h}W). \tag{4.59}$$

Since the characteristic vector field ξ is harmonic vector field for the paracontact $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifold it is enough to calculate (1.5).

Case 1: Assume that $\tilde{\kappa} > -1$.

Using (4.2), (4.3) and (4.59) in (1.5), we get

$$tr[R(\nabla.\xi, \xi).] = (\tilde{\lambda} - 1)\tilde{R}(\xi, \tilde{\varphi}\tilde{e})\tilde{e} + (\tilde{\lambda} + 1)\tilde{R}(\xi, \tilde{e})\tilde{\varphi}\tilde{e} = 2\tilde{\lambda^2}\tilde{\nu}\xi.$$

So, we conclude that $tr[R(\nabla.\xi, \xi).] = 0$ if and only if $\tilde{\nu} = 0$.

Case 2: We suppose that $\tilde{\kappa} = -1$. In this case we know that $\tilde{\nu}$ vanishes.

By help of (4.41), (4.42), (4.43) and (4.59), we obtain directly $tr[R(\nabla.\xi, \xi).] = 0$.

Case 3: Now we consider $\tilde{\kappa} < -1$.

Using (4.28), (4.29) and (4.59) in (1.5), we have

$$tr[R(\nabla.\xi, \xi).] = -2\tilde{\lambda}^2\tilde{\nu}\xi.$$

Therefore, we deduce that $tr[R(\nabla.\xi, \xi).] = 0$ if and only if $\tilde{\nu} = 0$.

Thus, we complete the proof of the theorem. □

5. An Application

Now we will give some properties of 3-dimensional contact metric manifolds.

Let $(M, \varphi, \xi, \eta, g)$ be a contact metric 3-manifold. Let

$$\begin{aligned} U &= \{p \in M \mid h(p) \neq 0\} \subset M, \\ U_0 &= \{p \in M \mid h(p) = 0, \text{ in a neighborhood of p}\} \subset M. \end{aligned}$$

That h is a smooth function on M implies $U \cup U_0$ is an open and dense subset of M, so any property satisfied in $U_0 \cup U$ is also satisfied in M. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \varphi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (this we call a φ-basis). On U, we put $he = \lambda e$, $h\varphi e = -\lambda\varphi e$, where λ is a nonvanishing smooth function assumed to be positive.

Lemma 5.1 ([17])**.** *(see also* [10]*) On the open set U we have*

$$\nabla_\xi e = a\varphi e,\ \nabla_e e = b\varphi e,\ \nabla_{\varphi e} e = -c\varphi e + (\lambda - 1)\xi, \tag{5.1}$$

$$\nabla_\xi \varphi e = -ae, \nabla_e \varphi e = -be + (1 + \lambda)\xi, \nabla_{\varphi e}\varphi e = ce, \tag{5.2}$$

$$\nabla_\xi \xi = 0,\ \nabla_e \xi = -(1 + \lambda)\varphi e,\ \nabla_{\varphi e}\xi = (1 - \lambda)e, \tag{5.3}$$

$$\nabla_\xi h = -2ah\varphi + \xi(\lambda)s, \tag{5.4}$$

where a is a smooth function,

$$b = \frac{1}{2\lambda}(\varphi e(\lambda) + A) \quad \text{with} \quad A = \eta(Qe) = S(\xi, e), \tag{5.5}$$

$$c = \frac{1}{2\lambda}(e(\lambda) + B) \quad \text{with} \quad B = \eta(Q\varphi e) = S(\xi, \varphi e), \tag{5.6}$$

and s is the type $(1,1)$ tensor field defined by $s\xi = 0$, $se = e$ and $s\varphi e = -\varphi e$.

In [4], Boeckx provided a local classification of non-Sasakian (κ,μ)-contact metric manifold respect to the number

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}}, \tag{5.7}$$

which is an invariant of a (κ,μ)-contact metric manifold up to $\mathcal{D}_\alpha$-homothetic deformations.

For (κ,μ,ν)-contact metric manifolds, T. Koufogiorgos et al. [20] proved that the following relations hold

$$h^2 = (\kappa - 1)\varphi^2 \text{ for } \kappa \leq 1, \tag{5.8}$$

$$\xi(\kappa) = 2\nu(\kappa - 1), \quad \xi(\lambda) = \nu(\lambda), \tag{5.9}$$

$$\nabla_\xi h = \mu h\varphi + \nu h. \tag{5.10}$$

Recently, in [19], the authors gave the following local classification of a non-Sasakian (κ,μ,ν)-contact metric manifolds with $\xi(I_M) = 0$.

Theorem 5.2 ([19]). *Let (M,φ,ξ,η,g) be a non-Sasakian $(\kappa,\mu,\nu = const.)$-contact metric manifold and $\xi(I_M) = 0$, where $\nu = const. \neq 0$. Then*

1) At any point of M, precisely one of the following relations is valid: $\mu = 2(1+\sqrt{1-\kappa})$, or $\mu = 2(1-\sqrt{1-\kappa})$

2) At any point $p \in M$ there exists a chart $(U,(x,y,z))$ with $p \in U \subseteq M$, such that

i) the functions κ, μ depend only on the variables x, z.

ii) if $\mu = 2(1+\sqrt{1-\kappa})$, (resp. $\mu = 2(1-\sqrt{1-\kappa})$), the tensor fields η, ξ, φ, g, h are given by the relations,

$$\xi = \frac{\partial}{\partial x}, \quad \eta = dx - adz,$$

$$g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1+a^2+b^2 \end{pmatrix} \quad \left(\text{resp.} \quad g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1+a^2+b^2 \end{pmatrix}\right),$$

$$\varphi = \begin{pmatrix} 0 & a & -ab \\ 0 & b & -1-b^2 \\ 0 & 1 & -b \end{pmatrix} \quad \left(\text{resp.} \quad \varphi = \begin{pmatrix} 0 & -a & ab \\ 0 & -b & 1+b^2 \\ 0 & -1 & b \end{pmatrix}\right),$$

$$h = \begin{pmatrix} 0 & 0 & -a\lambda \\ 0 & \lambda & -2\lambda b \\ 0 & 0 & -\lambda \end{pmatrix} \quad \left(\text{resp.} \quad h = \begin{pmatrix} 0 & 0 & a\lambda \\ 0 & -\lambda & 2\lambda b \\ 0 & 0 & \lambda \end{pmatrix}\right),$$

with respect to the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, where $a = 2y + f(z)$ (resp. $a = -2y + f(z)$), $b = -\frac{y^2}{2}\nu - y\frac{f(z)}{2}\nu - \frac{y}{2}\frac{r'(z)}{r(z)} + \frac{2}{\nu}r(z)e^{\nu x} + s(z)$ (resp. $b = \frac{y^2}{2}\nu - y\frac{f(z)}{2}\nu - \frac{y}{2}\frac{r'(z)}{r(z)} + \frac{2}{\nu}r(z)e^{\nu x} + s(z)$), $\lambda = \lambda(x,z) = r(z)e^{\nu x}$ and $f(z)$, $r(z)$, $s(z)$ are arbitrary smooth functions of z.

Now we are going to give a natural relation between non-Sasakian (κ,μ,ν)-contact metric manifolds with $\xi(I_M) = 0$ and 3-dimensional paracontact metric manifolds.

If Theorem 2.1 is adapted for the 3-dimensional non-Sasakain (κ,μ,ν)-contact metric manifold and used same procedure for proof then we have same result. Hence we can give following theorem.

Theorem 5.3 ([11])**.** *Let (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold. Then M admits a canonical paracontact metric structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$ is given by*

$$\tilde{\varphi} := \frac{1}{\sqrt{1-\kappa}}h, \quad \tilde{g} := \frac{1}{\sqrt{1-\kappa}}d\eta(\cdot,h\cdot)+\eta\otimes\eta. \tag{5.11}$$

After a long but straightforward calculation as in [11], we get

Proposition 5.4. *Let (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold. Then the Levi-Civita connections ∇ and $\tilde{\nabla}$ of g and $\tilde{g}$ are related as*

$$\begin{aligned}
\tilde{\nabla}_X Y \quad = \quad & \nabla_X Y + \frac{1}{2(1-\kappa)}\varphi h(\nabla_X\varphi h)Y - \frac{1}{\sqrt{1-\kappa}}\eta(Y)hX - \frac{1}{\sqrt{1-\kappa}}\eta(X)hY \\
& -\frac{1}{2}\eta(Y)\varphi hX - \frac{(1-\mu)}{2}\eta(Y)\varphi X - \frac{\nu}{2}\eta(Y)\varphi^2 X \\
& +(\frac{1}{2\sqrt{1-\kappa}}g(hX,Y)+\sqrt{1-\kappa}g(X,Y)-\sqrt{1-\kappa}\eta(X)\eta(Y) \\
& +\frac{(1-\mu)}{2\sqrt{1-\kappa}}g(hX,Y)-g(X,\varphi Y)+X(\eta(Y))-\eta(\nabla_X Y))\xi \\
& -\frac{1}{2}(1-\kappa)(X(\frac{1}{\sqrt{1-\kappa}})\varphi^2 Y+Y(\frac{1}{\sqrt{1-\kappa}})\varphi^2 X+ \\
& +\frac{1}{(1-\kappa)}g(X,\varphi hY)\varphi h grad(\frac{1}{\sqrt{1-\kappa}})),
\end{aligned} \tag{5.12}$$

for any $X,Y\in\Gamma(TM)$.

Now we will give a relation between $\tilde{h}$ and h by following lemma.

Lemma 5.5. *Let (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold and let $(\tilde{\varphi},\xi,\eta,\tilde{g})$ be the canonical paracontact metric structure induced on M, according to Theorem 5.3. Then we have*

$$\tilde{h} = \frac{1}{2\sqrt{1-\kappa}}((2-\mu)\varphi\circ h+2(1-\kappa)\varphi). \tag{5.13}$$

Proof. By help the equations (5.9), (5.11) and the definitons of $\tilde{h}$ and h we have

$$\begin{aligned}
2\tilde{h} \quad &= \quad L_\xi\tilde{\varphi} = L_\xi(\frac{1}{\sqrt{1-\kappa}}h) \\
&= \quad -\frac{\nu}{\sqrt{1-\kappa}}h+\frac{1}{2\sqrt{1-\kappa}}L_\xi(L_\xi\varphi).
\end{aligned} \tag{5.14}$$

Using the identities $\nabla\xi=-\varphi-\varphi h, \nabla_\xi\varphi=0$ and $\varphi^2 h=-h$, and following same procedure cf. [[11] pg. 270], we get

$$2\tilde{h} = -\frac{\nu}{\sqrt{1-\kappa}}h+\frac{1}{2\sqrt{1-\kappa}}(2\nabla_\xi h+4h^2\varphi-4h\varphi).$$

By using (5.8) and (5.10), we obtain the claimed relation. □

Let (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold. Choosing a local orthonormal φ-basis $\{e,\varphi e,\xi\}$ on (M,φ,ξ,η,g) and using Proposition 5.4, Lemma 5.5 and Lemma 5.1, one can give following proposition.

Proposition 5.6. *Let (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold and let $(\tilde{\varphi},\xi,\eta,\tilde{g})$ be the canonical paracontact metric structure induced on M, according to Theorem 5.3. Then we have*

$$\begin{aligned}
i)\ \tilde{\nabla}_e e &= -\frac{1}{2\lambda}e(\lambda)e+(\lambda+1-\frac{\mu}{2})\xi \quad ii)\ \tilde{\nabla}_e\varphi e=\frac{1}{2\lambda}e(\lambda)\varphi e+\xi, \quad iii)\ \tilde{\nabla}_e\xi=-e+(\frac{\mu}{2}-1-\lambda)\varphi e,\\
iv)\ \tilde{\nabla}_{\varphi e}e &= \frac{1}{2\lambda}(\varphi e)(\lambda)e-\xi, \quad v)\ \tilde{\nabla}_{\varphi e}\varphi e=-\frac{1}{2\lambda}(\varphi e)(\lambda)\varphi e+(\lambda-1+\frac{\mu}{2})\xi,\\
vi)\ \tilde{\nabla}_{\varphi e}\xi &= -e-(\lambda-1+\frac{\mu}{2})\varphi e,\\
vii)\ \tilde{\nabla}_\xi e &= -e, \quad viii)\ \tilde{\nabla}_\xi\varphi e=\varphi e,\\
ix)\ [e,\xi] &= (\frac{\mu}{2}-1-\lambda)\varphi e, \ x)\ [\varphi e,\xi]=-e-(\lambda+\frac{\mu}{2})\varphi e, \quad xi)\ [e,\varphi e]=-\frac{1}{2\lambda}(\varphi e)(\lambda)e+\frac{1}{2\lambda}e(\lambda)\varphi e+2\xi.
\end{aligned}$$

Moreover, $\tilde{g}(e,e)=\tilde{g}(\varphi e,\varphi e)=\tilde{g}(\varphi e,\xi)=0$ and $\tilde{g}(e,\varphi e)=\tilde{g}(\xi,\xi)=1$.

We assume that (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold, $(\tilde{\varphi},\xi,\eta,\tilde{g})$ be the canonical paracontact metric structure induced on M, according to Theorem 5.3. Let $\{e,\varphi e,\xi\}$ be an orthonormal φ-basis in a neighborhood of $p\in M$. Then one can always construct an orthonormal $\tilde{\varphi}$-basis $\{\tilde{e}_1,\tilde{\varphi}\tilde{e}_1=\tilde{e}_2,\xi\}$, for instance $\tilde{e}_1=(e-\varphi e)/\sqrt{2}$, $\tilde{\varphi}\tilde{e}_1=(e+\varphi e)/\sqrt{2}$, such that $\tilde{g}(\tilde{e}_1,\tilde{e}_1)=-1$, $\tilde{g}(\tilde{\varphi}\tilde{e}_1,\tilde{\varphi}\tilde{e}_1)=1$, $\tilde{g}(\xi,\xi)=1$. Moreover, from Lemma 5.5, the matrix form of $\tilde{h}$ is given by

$$\tilde{h}=\begin{pmatrix} -1+\frac{\mu}{2} & -\lambda & 0\\ \lambda & 1-\frac{\mu}{2} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{5.15}$$

with respect to local orthonormal basis $\{\tilde{e}_1,\tilde{\varphi}\tilde{e}_1,\xi\}$. By using Proposition 5.6 we have following proposition.

Proposition 5.7. *Let (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold and let $(\tilde{\varphi},\xi,\eta,\tilde{g})$ be the canonical paracontact metric structure induced on M, according to Theorem 5.3. Then we have*

$$\begin{aligned}
i)\ \tilde{\nabla}_{\tilde{e}_1}\tilde{e}_1 &= -\frac{1}{2\lambda}(\tilde{\varphi}\tilde{e}_1)(\lambda)\tilde{\varphi}\tilde{e}_1+\lambda\xi, \quad ii)\ \tilde{\nabla}_{\tilde{e}_1}\tilde{\varphi}\tilde{e}_1=-\frac{1}{2\lambda}(\tilde{\varphi}\tilde{e}_1)(\lambda)\tilde{e}_1+(2-\frac{\mu}{2})\xi,\\
iii)\ \tilde{\nabla}_{\tilde{e}_1}\xi &= \lambda\tilde{e}_1+(\frac{\mu}{2}-2)\tilde{\varphi}\tilde{e}_1, \quad iv)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}_1}\tilde{e}_1=-\frac{1}{2\lambda}\tilde{e}_1(\lambda)\tilde{\varphi}\tilde{e}_1-\frac{\mu}{2}\xi,\\
v)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}_1}\tilde{\varphi}\tilde{e}_1 &= -\frac{1}{2\lambda}\tilde{e}_1(\lambda)\tilde{e}_1+\lambda\xi, \quad vi)\ \tilde{\nabla}_{\tilde{\varphi}\tilde{e}_1}\xi=-\frac{\mu}{2}\tilde{e}_1-\lambda\tilde{\varphi}\tilde{e}_1,\\
vii)\ \tilde{\nabla}_\xi\tilde{e}_1 &= -\tilde{\varphi}\tilde{e}_1, \quad viii)\ \tilde{\nabla}_\xi\tilde{\varphi}\tilde{e}_1=-\tilde{e}_1,\\
ix)\ [\tilde{e}_1,\xi] &= \lambda\tilde{e}_1+(\frac{\mu}{2}-1)\tilde{\varphi}\tilde{e}_1 \quad x)\ [\tilde{\varphi}\tilde{e}_1,\xi]=(1-\frac{\mu}{2})\tilde{e}_1-\lambda\tilde{\varphi}\tilde{e}_1,\\
xi)\ [\tilde{e}_1,\tilde{\varphi}\tilde{e}_1] &= -\frac{1}{2\lambda}(\tilde{\varphi}\tilde{e}_1)(\lambda)\tilde{e}_1+\frac{1}{2\lambda}\tilde{e}_1(\lambda)\tilde{\varphi}\tilde{e}_1+2\xi. \qquad (5.16)
\end{aligned}$$

Proposition 5.8. *Let (M,φ,ξ,η,g) be a non-Sasakian $(\kappa,\mu,\nu=const.)$-contact metric manifold with $\xi(I_M)=0$ and suppose that $(\tilde{\varphi},\xi,\eta,\tilde{g})$ be the canonical paracontact metric structure induced on M. Then the following equation holds*

$$\tilde{\nabla}_\xi\tilde{h}=2\tilde{h}\tilde{\varphi}+\nu\tilde{h}. \tag{5.17}$$

Proof. Taking $\xi(\mu)=\nu(\mu-2)$ and $\xi(\lambda)=\nu\lambda$ into account and using the relations (5.15), (5.16) we get requested relation. $\square$

We suppose that (M,φ,ξ,η,g) be a non-Sasakian (κ,μ,ν)-contact metric manifold with $\xi(I_M)=0$. Let $(\tilde{\varphi},\xi,\eta,\tilde{g})$ be the canonical paracontact metric structure induced on M, according to Theorem 5.3.

By (2.4) and (5.16), and after very long computations we obtain that

$$\tilde{R}(\tilde{e}_1, \tilde{\varphi}\tilde{e}_1)\xi = (\frac{1}{\lambda}(\frac{\mu}{2} - 1)\tilde{e}_1(\lambda) - \frac{1}{2}\tilde{e}_1(\mu))\tilde{e}_1 + (\frac{1}{\lambda}(\frac{\mu}{2} - 1)(\tilde{\varphi}\tilde{e}_1)(\lambda) - \frac{1}{2}(\tilde{\varphi}\tilde{e}_1)(\mu))\tilde{\varphi}\tilde{e}_1. \tag{5.18}$$

By using Theorem 5.2 and $\tilde{e}_1 = (e - \varphi e)/\sqrt{2}, \tilde{\varphi}\tilde{e}_1 = (e + \varphi e)/\sqrt{2}$ in (5.18) one can check that in fact $\tilde{R}(\tilde{e}_1, \tilde{\varphi}\tilde{e}_1)\xi = 0$. Now if we use (4.1), we have

$$\tilde{R}(\tilde{e}_1, \tilde{\varphi}\tilde{e}_1)\xi = -\tilde{\sigma}(\tilde{e}_1)\tilde{\varphi}\tilde{e}_1 + \tilde{\sigma}(\tilde{\varphi}\tilde{e}_1)\tilde{e}_1. \tag{5.19}$$

Comparing $\tilde{R}(\tilde{e}_1, \tilde{\varphi}\tilde{e}_1)\xi = 0$ with (5.19), we get

$$\tilde{\sigma}(\tilde{e}_1) = \tilde{\sigma}(\tilde{\varphi}\tilde{e}_1) = 0. \tag{5.20}$$

So ξ is an eigenvector of Ricci operator $\tilde{Q}$.

Remark 5.9. From (1.3) and (5.16) we have

$$\begin{aligned} \tilde{\nabla}^*\tilde{\nabla}\xi &= -\tilde{\nabla}_{\tilde{e}_1}\tilde{\nabla}_{\tilde{e}_1}\xi + \tilde{\nabla}_{\tilde{\nabla}_{\tilde{e}_1}\tilde{e}_1}\xi + \tilde{\nabla}_{\tilde{\varphi}\tilde{e}_1}\tilde{\nabla}_{\tilde{\varphi}\tilde{e}_1}\xi - \nabla_{\nabla_{\tilde{\varphi}\tilde{e}_1}\tilde{\varphi}\tilde{e}_1}\xi \\ &= (\frac{1}{\lambda}(\frac{\mu}{2} - 1)(\tilde{\varphi}\tilde{e}_1)(\lambda) - \frac{1}{2}(\tilde{\varphi}\tilde{e}_1)(\mu))\tilde{e}_1 \\ &\quad +(\frac{1}{\lambda}(\frac{\mu}{2} - 1)\tilde{e}_1(\lambda) - \frac{1}{2}\tilde{e}_1(\mu))\tilde{\varphi}\tilde{e}_1 \\ &\quad +((2 - \frac{\mu}{2})^2 + \frac{\mu^2}{4} - 2\lambda^2)\xi. \end{aligned} \tag{5.21}$$

Then by using again Theorem 5.2 and $\tilde{e}_1 = (e - \varphi e)/\sqrt{2}, \tilde{\varphi}\tilde{e}_1 = (e + \varphi e)/\sqrt{2}$ in (5.21) and after long computations one can prove that the last relation reduces to

$$\tilde{\nabla}^*\tilde{\nabla}\xi = 2\xi. \tag{5.22}$$

By virtue of (5.22), we see immediately that ξ is harmonic vector field on $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$.

So we can give following theorem:

Theorem 5.10. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian $(\kappa, \mu, \nu = const)$-contact metric manifold with $\xi(I_M) = 0$ and let $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ be the canonical paracontact metric structure induced on M, according to Theorem 5.3. Then the characteristic vector field ξ is an eigenvector of the Ricci operator .*

By using previous theorem and same procedure as in Case 3 of the proof of Theorem 1.2, we obtain following theorem.

Theorem 5.11. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian $(\kappa, \mu, \nu = const)$-contact metric manifold with $\xi(I_M) = 0$ and let $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ be the canonical paracontact metric structure induced on M, according to Theorem 5.3. Then the curvature tensor field of the Levi Civita connection of $(M, \tilde{g})$ verifies the following relation*

$$\tilde{R}(X, Y)\xi = (\kappa - 2)(\eta(Y)X - \eta(X)Y) + 2(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) - \nu(\eta(Y)\tilde{\varphi}\tilde{h}X - \eta(X)\tilde{\varphi}\tilde{h}Y).$$

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