Hankel Inequalities for a Subclass of Bi-Univalent Functions based on Salagean type q-Difference Operator

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Abstract. In this investigation a new subclass of bi-univalent functions is established that are defined in the open unit disk $\Delta = \{\mathcal{Z} \in \mathbb{C} : |\mathcal{Z}| < 1\}$ and are endowed with the Sălăgean type q-difference operator. Then, Hankel inequalities for the new function class are obtained and several related consequences of the results are also stated.

Keywords: *bi-univalent; coefficient bounds; convex functions; Hankel inequalities; Starlike; univalent.*

1 Introduction

Let $\mathcal{A}$ indicate the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ indicate the subclass of $\mathcal{A}$ comprising of functions of the form Eq. (1) and also univalent in Δ.

For the function $f \in \mathcal{A}$, Jackson's q-derivative [1] $(0 < q < 1)$ is expressed by:

$$\mathfrak{D}_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \tag{2}$$

and $\mathfrak{D}_q^2 f(z) = \mathfrak{D}_q(\mathfrak{D}_q f(z))$. Thus, from Eq. (2), we deduce that

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

If $q \to 1^-$, we get $[n]_q \to n$.

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Lately, in [2] the Sălăgean type q-differential operator has been introduced as given by

$$\mathfrak{D}_q^0 f(z) = f(z)$$

$$\mathfrak{D}_q^1 f(z) = z\mathfrak{D}_q f(z)$$

$$\mathfrak{D}_q^k f(z) = z\mathfrak{D}_q(\mathfrak{D}_q^{k-1} f(z))$$

$$\mathfrak{D}_q^k f(z) = z + \sum_{n=2}^{\infty} [n]_q^k a_n z^n \quad (k \in \mathbb{N}_0, z \in \Delta). \tag{3}$$

For $q \to 1^-$, we get

$$\mathfrak{D}^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (k \in \mathbb{N}_0, z \in \Delta)$$

the familiar Sălăgean derivative [3].

Noonan and Thomas [4] introduced the $\boldsymbol{q}^{th}$ Hankel determinant of function f by

$$H_{\boldsymbol{q}}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\boldsymbol{q}-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\boldsymbol{q}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+\boldsymbol{q}-1} & a_{n+\boldsymbol{q}} & \cdots & a_{n+2\boldsymbol{q}-2} \end{vmatrix} \quad (\boldsymbol{q} \geq 1).$$

In particular,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1 a_3 - a_2^2 = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

Then, Fekete and Szegö [5] obtained estimates of $|H_2(1)| = |a_3 - \theta a_2^2|$ for θ is real. That is, if $f \in \mathcal{A}$, then

$$|a_3 - \theta a_2^2| \leq \begin{cases} 4\theta - 3 & \theta \geq 1 \\ 1 + 2\, exp(\frac{-2\theta}{1-\theta}) & 0 \leq \theta \leq 1. \\ 3 - 4\theta & \theta \leq 0 \end{cases}$$

Furthermore, Keogh and Merkes [6] derived sharp estimates for $|H_2(1)|$ when f is starlike, convex and close-to-convex in Δ.

Next, according to the Koebe One Quarter Theorem [7], every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, (z \in \Delta)$ and $f(f^{-1}(w)) = w$ $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$. A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ indicate the class of bi-univalent functions defined in the unit disk Δ. Since $f \in \Sigma$ has the Taylor representation given by Eq. (1), computation shows that $g = f^{-1}$ has the following representation:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots. \tag{4}$$

Several researchers have introduced new subclasses of bi-univalent functions and derived non-sharp the initial coefficients (see [8-18]).

Now, by using the Sălăgean type q-differential operator for functions g of the form Eq. (4), we define:

$$\mathfrak{D}_q^k g(w) = w - a_2[2]_q^k w^2 + (2a_2^2 - a_3)[3]_q^k w^3 + \cdots \tag{5}$$

and introduce a new subclass of Σ to acquire the estimates of the initial Taylor-Maclaurin coefficients. Then, by using the values of a_2 and a_3, we derive the Fekete-Szegö and Hankel inequalities.

2 Bi-Univalent Function Class $\mathcal{F}\Sigma_q^k(\lambda, \beta)$

In this section, we will give the following new subclass involving the Sălăgean type q-difference operator and also its related classes.

Definition 2.1. A function $f \in \Sigma$ given by Eq. (1) is said to be in the class

$$\mathcal{F}\Sigma_q^k(\lambda, \beta) \qquad (0 \le \beta < 1, 0 \le \lambda \le 1, z, w \in \Delta)$$

if the following conditions hold:

$$\Re\left((1-\lambda)\frac{\mathfrak{D}_q^k f(z)}{z} + \lambda\left(\mathfrak{D}_q^k f(z)\right)'\right) > \beta$$

and

$$\Re\left((1-\lambda)\frac{\mathfrak{D}_q^k g(w)}{w} + \lambda\left(\mathfrak{D}_q^k g(w)\right)'\right) > \beta.$$

Example 2.2. A function $f \in \Sigma$, members of which are given by Eq. (1) and

1. for $\lambda = 0$, let $\mathcal{F}\Sigma_q^k(0, \beta) =: \mathcal{R}\Sigma_q^k(\beta)$ denote the subclass of Σ and the following conditions hold

$$\Re\left(\frac{\mathfrak{D}_q^k f(z)}{z}\right) > \beta \qquad \text{and} \qquad \Re\left(\frac{\mathfrak{D}_q^k g(w)}{w}\right) > \beta$$

2. for $\lambda = 1$, let $\mathcal{F}\Sigma_q^k(1, \beta) =: \mathcal{H}\Sigma_q^k(\beta)$ denote the subclass of Σ and satisfy the following conditions

$$\Re\left[\left(\mathfrak{D}_q^k f(z)\right)'\right] > \beta \qquad \text{and} \qquad \Re\left[\left(\mathfrak{D}_q^k g(w)\right)'\right] > \beta.$$

3 Hankel Inequalities for $f \in \mathcal{F}\Sigma_q^k(\lambda, \beta)$

In this section, we will determine the functional $|a_2 a_4 - a_3^2|$ for the functions $f \in \mathcal{F}\Sigma_{\mathrm{q}}^k(\lambda, \beta)$ due to Altınkaya and Yalçın [19]. Now, we recall the following lemmas:

Lemma 3.1. (See [4]) Let $\mathcal{P}$ be the well-known class of Carathéodory functions, that is $c(z) \in \mathcal{A}$ with the power series expansion

$$c(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \Delta) \tag{6}$$

and $\Re(c(z)) > 0$. Then

$$|c_n| \leq 2 \ \ (n = 1,2,3,\dots)$$

and is sharp for each n. Indeed,

$$c(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2 z^n \qquad (\forall n \geq 1).$$

Lemma 3.2. (See [20]) If $c \in \mathcal{P}$, then

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{7}$$
$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2 z)$$

for some complex numbers x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 3.3. (See [5]) The power series for c converges in Δ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants

$$T_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix} \quad (\ n = 1,2,3, \dots)$$

and $c_{-\kappa} = \overline{c_\kappa}$ are all nonnegative. They are exactly positive except for

$$c(z) = \sum_{\kappa=1}^{m} \rho_\kappa \, c_0(e^{it_\kappa z}), \ \ \rho_\kappa > 0, \ \ t_\kappa \text{ real}$$

and $t_\kappa \neq t_j$ $(\kappa \neq j)$. In this case $T_n > 0$ $(n < m - 1)$ and $T_n = 0$ $(n \geq m)$.

Next, we designate the second Hankel coefficient estimates for $f \in \mathcal{F}\Sigma_q^k(\lambda, \beta)$.

Theorem 3.4. Let $f \in \mathcal{F}\Sigma_q^k(\lambda,\beta)$. Then

$$|a_2a_4 - a_3^2| \leq \begin{cases} H(2), & A(\beta,\lambda,k,q) \geq 0, B(\beta,\lambda,k,q) \geq 0 \\ \max\left\{\dfrac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}, H(2)\right\}, & A(\beta,\lambda,k,q) > 0, B(\beta,\lambda,k,q) < 0 \\ \dfrac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}, & A(\beta,\lambda,k,q) \leq 0, B(\beta,\lambda,k,q) \leq 0 \\ \max\{H(\varepsilon_0), H(2)\}, & A(\beta,\lambda,k,q) < 0, B(\beta,\lambda,k,q) > 0 \end{cases},$$

where

$$H(2) = \frac{16(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} + \frac{4(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k},$$

$$H\left(\varepsilon_0 = \sqrt{\frac{-B(\beta,\lambda,k,q)}{A(\beta,\lambda,k,q)}}\right) = \frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}} - \frac{B^2(\beta,\lambda,k,q)}{4A(\beta,\lambda,k,q)},$$

$$A(\beta,\lambda,k,q) = \frac{(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} - \frac{(1-\beta)^3}{4(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k} - \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k} + \frac{(1-\beta)^2}{4(1+2\lambda)^2[3]_q^{2k}}$$

$$B(\beta,\lambda,k,q) = \frac{(1-\beta)^3}{(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k} + \frac{3(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k} - \frac{2(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}.$$

Proof. Suppose that $f \in \mathcal{F}\Sigma_q^k(\beta,\lambda)$. There are two functions $\phi, \psi \in \mathcal{P}$ satisfying the conditions of Lemma 3.1 such that

$$(1-\lambda)\frac{\mathfrak{D}_q^k f(z)}{z} + \lambda\left(\mathfrak{D}_q^k f(z)\right)' = \beta + (1-\beta)\phi(z), \tag{8}$$

$$(1-\lambda)\frac{\mathfrak{D}_q^k g(w)}{w} + \lambda\left(\mathfrak{D}_q^k g(w)\right)' = \beta + (1-\beta)\psi(z), \tag{9}$$

where

$$\phi(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots,$$

$$\psi(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots.$$

Now, by comparing the corresponding coefficients in Eq. (8) and Eq. (9), we get

$$(1+\lambda)[2]_q^k a_2 = (1-\beta)c_1, \tag{10}$$

$$(1+2\lambda)[3]_q^k a_3 = (1-\beta)c_2, \tag{11}$$

$$(1+3\lambda)[4]_q^k a_4 = (1-\beta)c_3 \tag{12}$$

and

$$-(1+\lambda)[2]_q^k a_2 = (1-\beta)d_1, \tag{13}$$

$$(1+2\lambda)[3]_q^k (2a_2^2 - a_3) = (1-\beta)d_2, \tag{14}$$

$$-(1+3\lambda)[4]_q^k (5a_2^3 - 5a_2 a_3 + a_4) = (1-\beta)d_3. \tag{15}$$

From Eq. (10) and Eq. (13), we get

$$a_2 = \frac{1-\beta}{(1+\lambda)[2]_q^k} c_1 = -\frac{1-\beta}{(1+\lambda)[2]_q^k} d_1, \tag{16}$$

which implies

$$c_1 = -d_1.$$

Now from Eq. (11) and Eq. (14), we obtain

$$a_3 = \frac{(1-\beta)^2}{(1+\lambda)^2\,[2]_q^{2k}}\, c_1^2 + \frac{(1-\beta)}{2(1+2\lambda)[3]_q^k}(c_2 - d_2).$$

On the other hand, subtracting Eq. (15) from Eq. (12) and using Eq. (16), we get

$$a_4 = \frac{5(1-\beta)^2}{4(1+\lambda)(1+2\lambda)[2]_q^k[3]_q^k} c_1(c_2 - d_2) + \frac{(1-\beta)}{2(1+3\lambda)[4]_q^k}(c_3 - d_3).$$

Thus, we establish that

$$|a_2 a_4 - a_3^2| = \left| \begin{array}{c} -\frac{(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} c_1^4 \\ + \frac{(1-\beta)^3}{4(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k} c_1^2(c_2 - d_2) \end{array} \right.$$

$$\left. + \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k} c_1(c_3 - d_3) - \frac{(1-\beta)^2}{4(1+2\lambda)[3]_q^{2k}}(c_2 - d_2)^2 \right|. \tag{17}$$

Now, by Lemma 3.2, we get

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad \text{and } 2d_2 = d_1^2 + y(4 - d_1^2), \tag{18}$$

and hence, by Eq. (18), we have

$$c_2 - d_2 = \frac{4-c_1^2}{2}(x-y). \tag{19}$$

Further, we get

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z,$$

$$4d_3 = d_1^3 + 2(4-d_1^2)d_1y - d_1(4-d_1^2)y^2 + 2(4-d_1^2)(1-|y|^2)w$$

and thus, we acquire

$$c_3 - d_3 = \frac{c_1^3}{2} + \frac{c_1(4-c_1^2)}{2}(x+y) - \frac{c_1(4-c_1^2)}{4}(x^2+y^2) \tag{20}$$
$$+\frac{4-c_1^2}{2}[(1-|x|^2)z - (1-|y|^2)w].$$

Using Eq. (19) – Eq. (20) in Eq. (17), we get

$$|a_2a_4 - a_3^2| =$$
$$\left|\frac{-(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}}c_1^4 + \frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}c_1^4\right.$$
$$+ \frac{(1-\beta)^3}{4(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k}\frac{c_1^2(4-c_1^2)}{2}(x-y)$$
$$+\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\frac{c_1^2(4-c_1^2)}{2}(x+y)$$
$$-\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\frac{c_1^2(4-c_1^2)}{4}(x^2+y^2)$$
$$= + \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\frac{c_1(4-c_1^2)}{2}[(1-|x|^2)z - (1-|y|^2)w]$$
$$\left.-\frac{(1-\beta)^2}{4(1+2\lambda)^2[3]_q^{2k}}\frac{(4-c_1^2)^2}{4}(x-y)^2\right|.$$

Since $c \in \mathcal{P}$, we find that $|c_1| \leq 2$. Thus, letting $|c_1| = \varepsilon \in [0,2]$ and applying triangle inequality on Eq. (21), we get

$$|a_2a_4 - a_3^2| \leq \frac{(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}}\varepsilon^4 + \frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\varepsilon^4$$
$$+\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\varepsilon(4-\varepsilon^2)$$
$$+\left(\frac{(1-\beta)^3}{4(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k} + \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\right)\frac{\varepsilon^2(4-\varepsilon^2)}{2}(|x|+|y|)$$
$$+ \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\frac{\varepsilon(\varepsilon-2)(4-\varepsilon^2)}{4}(|x|^2+|y|^2)$$

$$-\frac{(1-\beta)^2}{4(1+2\lambda)^2[3]_q^{2k}}\frac{(4-\varepsilon^2)^2}{4}(|x|+|y|)^2. \tag{21}$$

For $\delta=|x|\leq 1$ and $\vartheta=|y|\leq 1$, we get

$$|a_2a_4-a_3^2|\leq C_1+C_2(\delta+\vartheta)+C_3(\delta^2+\vartheta^2) \tag{22}$$
$$C_4(\delta+\vartheta)^2=\Psi(\delta,\vartheta),$$

where

$$\begin{aligned}
C_1=C_1(\varepsilon)&=\frac{(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}}\varepsilon^4+\frac{(1-\beta)^2}{4(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\varepsilon^4\\
&\quad+\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\varepsilon(4-\varepsilon^2)\ \geq 0,\\
C_2=C_2(\varepsilon)&=\Bigg(\frac{(1-\beta)^3}{4(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k}\\
&\quad+\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\Bigg)\frac{\varepsilon^2(4-\varepsilon^2)}{2}\geq 0,\\
C_3=C_3(\varepsilon)&=\frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\frac{\varepsilon(\varepsilon-2)(4-\varepsilon^2)}{4}\leq 0,\\
C_4=C_4(\varepsilon)&=\frac{(1-\beta)^2}{4(1+2\lambda)^2[3]_q^{2k}}\frac{(4-\varepsilon^2)^2}{4}\geq 0.
\end{aligned}$$

Next, we will find the maximum of $(\Psi(\delta,\vartheta))$ in $\Gamma=\{(\delta,\vartheta):0\leq\delta\leq 1,0\leq\vartheta\leq 1\}$. Since the coefficients of $\Psi(\delta,\vartheta)$ have dependent variable ε, we should maximize $\Psi(\delta,\vartheta)$ for the cases $\varepsilon=0,\varepsilon=2$ and $\varepsilon\in(0,2)$.

1. Let $\varepsilon=0$. Thus, from (22), we may write

$$\Psi(\delta,\vartheta)=\frac{(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}(\delta+\vartheta)^2.$$

2. We can find that the maximum of $\Psi(\delta,\vartheta)$ occurs at $\delta=\vartheta=1$ and we find

$$\max\{\Psi(\delta,\vartheta):0\leq\delta\leq 1,0\leq\vartheta\leq 1\}=\frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}.$$

3. Let $\varepsilon=2$. Thus, $\Psi(\delta,\vartheta)$ is a constant function

$$\Psi(\delta,\vartheta)=\frac{16(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}}+\frac{4(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}.$$

4. Let $\varepsilon\in(0,2)$. If we change $\delta+\vartheta=\zeta$ and $\delta.\vartheta=\eta$, then

$$\begin{aligned}
\Psi(\delta,\vartheta)&=C_1(\varepsilon)+C_2(\varepsilon)\zeta+[C_3(\varepsilon)+C_4(\varepsilon)]\zeta^2-2C_3(\varepsilon)\eta\\
&=\mathcal{G}(\zeta,\eta),\quad 0\leq\zeta\leq 2,0\leq\eta\leq 1.
\end{aligned}$$

Presently, we try to get maximum of $\mathcal{G}(\zeta,\eta)$ in

$$= \{(\zeta,\eta): 0 \leq \zeta \leq 2, 0 \leq \eta \leq 1\}.$$

From the definition of $\mathcal{G}(\zeta,\eta)$, we get

$$\mathcal{G}'_{\zeta}(\zeta,\eta) = C_2(\varepsilon) + 2[C_3(\varepsilon) + C_4(\varepsilon)]\zeta = 0,$$
$$\mathcal{G}'_{\zeta}(\zeta,\eta) = -2C_3(\varepsilon) = 0.$$

We deduce that the function doesn't have any critical point in . Thus, $\Psi(\delta,\vartheta)$ doesn't have any critical point in square Γ and so the function doesn't get maximum value in Γ.

Next, we inspect the maximum of $\Psi(\delta,\vartheta)$ on the boundary of Γ. Firstly, let $\delta = 0, 0 \leq \vartheta \leq 1$ (or let $\vartheta = 0, 0 \leq \delta \leq 1$). Then, we may write

$$\Psi(0,\vartheta) = C_1(\varepsilon) + C_2(\varepsilon)\vartheta + [C_3(\varepsilon) + C_4(\varepsilon)]\vartheta^2 = \varphi_1(\vartheta).$$

Thus,

$$\varphi_1'(\vartheta) = C_2(\varepsilon) + 2[C_3(\varepsilon) + C_4(\varepsilon)]\vartheta.$$

Case (i): If $C_3(\varepsilon) + C_4(\varepsilon) \geq 0$, then $\varphi_1'(\vartheta) > 0$. The function is increasing and so the maximum occurs at $\vartheta = 1$.

Case (ii): Let $C_3(\varepsilon) + C_4(\varepsilon) < 0$. Since $C_2(\varepsilon) + 2[C_3(\varepsilon) + C_4(\varepsilon)] > 0$, $C_2(\varepsilon) + 2[C_3(\varepsilon) + C_4(\varepsilon)]\vartheta \geq C_2(\varepsilon) + 2[C_3(\varepsilon) + C_4(\varepsilon)]$ holds for all $\vartheta \in [0,1]$. So, $\varphi_1'(\vartheta) > 0$. Hence, $\varphi_1(\vartheta)$ is an increasing function. Thus, the maximum occurs at $\vartheta = 1$,

$$\max\{\Psi(0,\vartheta): 0 \leq \vartheta \leq 1\} = C_1(\varepsilon) + C_2(\varepsilon) + C_3(\varepsilon) + C_4(\varepsilon).$$

Secondly, let $\delta = 1, 0 \leq \vartheta \leq 1$ (similarly, $\vartheta = 1, 0 \leq \delta \leq 1$). Then

$$\begin{aligned}\Psi(1,\vartheta) &= C_1(\varepsilon) + C_2(\varepsilon) + C_3(\varepsilon) + C_4(\varepsilon)\\ &+[C_2(\varepsilon) + 2C_4(\varepsilon)]\vartheta + [C_3(\varepsilon) + C_4(\varepsilon)]\vartheta^2\\ &= \varphi_2(\vartheta).\end{aligned}$$

It can be stated that $\varphi_2(\vartheta)$ is an increasing function like case (i). In that way,

$$\max\{\Psi(1,\vartheta): 0 \leq \vartheta \leq 1\} = C_1(\varepsilon) + 2[C_2(\varepsilon) + C_3(\varepsilon)] + 4C_4(\varepsilon).$$

Also, for every $\varepsilon \in (0,2)$, we can easily see that

$$C_1(\varepsilon) + 2[C_2(\varepsilon) + C_3(\varepsilon)] + 4C_4(\varepsilon) > C_1(\varepsilon) + C_2(\varepsilon) + C_3(\varepsilon) + C_4(\varepsilon).$$

Therefore, we find that

$$\max\{\Psi(\delta,\vartheta): 0 \leq \delta \leq 1, 0 \leq \vartheta \leq 1\} = C_1(\varepsilon) + 2[C_2(\varepsilon) + C_3(\varepsilon)] + 4C_4(\varepsilon).$$

Since $\varphi_1(1) \leq \varphi_2(1)$ for $\varepsilon \in [0,2]$, max $\Psi(\delta, \vartheta) = \Psi(1,1)$ on the boundary of Γ. So, the maximum of Ψ occurs at $\delta = 1$ and $\vartheta = 1$ in the Γ.

Let us define $\mathcal{H}: (0,2) \to \mathbb{R}$ as

$$\begin{aligned}\mathcal{H}(\varepsilon) &= \max\ \Psi(\delta, \vartheta) = \Psi(1,1)\\ &= 2[C_2(\varepsilon) + C_3(\varepsilon)] + C_1(\varepsilon) + 4C_4(\varepsilon).\end{aligned} \tag{23}$$

Therefore, from Eq. (23), we obtain

$$\mathcal{H}(\varepsilon) = \frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}} + A(\beta, \lambda, k, q)\ \varepsilon^4 + 2\ B(\beta, \lambda, k, q)\ \varepsilon^2,$$

where

$$\begin{aligned}A(\beta, \lambda, k, q) &= \frac{(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} - \frac{(1-\beta)^3}{4(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k}\\ &- \frac{(1-\beta)^2}{2(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k} + \frac{(1-\beta)^2}{4(1+2\lambda)^2[3]_q^{2k}}\end{aligned}$$

$$\begin{aligned}B(\beta, \lambda, k, q) &= \frac{(1-\beta)^3}{(1+\lambda)^2(1+2\lambda)[2]_q^{2k}[3]_q^k} + \frac{3(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\\ &- \frac{2(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}.\end{aligned}$$

Now, we try to get the maximum value of $\mathcal{H}(\varepsilon)$ in (0,2). After some basic calculations, we have

$$\mathcal{H}'(\varepsilon) = 4A(\beta, \lambda, k, q)\varepsilon^3 + 2B(\beta, \lambda, k, q)\varepsilon.$$

Next, we examine the different cases of $A(\beta, \lambda, k, q)$ and $B(\beta, \lambda, k, q)$ as follows:

Case 1: Let $A(\beta, \lambda, k, q) \geq 0$ and $B(\beta, \lambda, k, q) \geq 0$, then $\mathcal{H}'(\varepsilon) \geq 0$. Hence, the maximum point has to be on the boundary of $\varepsilon \in [0,2]$, that is $\varepsilon = 2$. Thus,

$$\begin{aligned}&\max\{\Psi(\delta, \vartheta): 0 \leq \delta \leq 1, 0 \leq \vartheta \leq 1\}\\ &= \mathcal{H}(2)\\ &= \frac{16(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} + \frac{4(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\end{aligned} \tag{24}$$

Case 2: If $A(\beta,\lambda,k,q) > 0$ and $B(\beta,\lambda,k,q) < 0, \varepsilon_0 = \sqrt{\frac{-B(\beta,\lambda,k,q)}{2A(\beta,\lambda,k,q)}}$ is a critical point of $\mathcal{H}(\varepsilon)$. Since $\mathcal{H}''(\varepsilon_0) < 0$, the maximum value of function $\mathcal{H}(\varepsilon)$ occurs at $\varepsilon = \varepsilon_0$ and

$$\mathcal{H}(\varepsilon_0) = \frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}} + A(\beta,\lambda,k,q)\,\varepsilon_0^4 + 2\,B(\beta,\lambda,k,q)\,\varepsilon_0^2$$
$$= \frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}} - \frac{3B^2(\beta,\lambda,k,q)}{4A(\beta,\lambda,k,q)}.$$

In this case, $\mathcal{H}(\varepsilon_0) < \frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}$. Therefore,

$$\max\{\Psi(\delta,\vartheta): 0 \le \delta \le 1, 0 \le \vartheta \le 1\}$$
$$= \max\left\{\frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}, \frac{16(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} + \frac{4(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\right\}. \tag{25}$$

Case 3: If $A(\beta,\lambda,k,q) \le 0$ and $B(\beta,\lambda,k,q) \le 0, \mathcal{H}(\varepsilon)$ is decreasing in (0,2). Therefore,

$$\max\{\Psi(\delta,\vartheta): 0 \le \delta \le 1, 0 \le \vartheta \le 1\} = \frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}} \tag{26}$$

Case 4: If $A(\beta,\lambda,k,q) < 0$ and $B(\beta,\lambda,k,q) > 0, \varepsilon_0$ is a critical point of $\mathcal{H}(\varepsilon)$. Since $\mathcal{H}''(\varepsilon_0) < 0$, the maximum value of $\mathcal{H}(\varepsilon)$ occurs at $\varepsilon = \varepsilon_0$ and

$$\frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}} < \mathcal{H}(\varepsilon_0).$$

Therefore,

$$\max\{\Psi(\delta,\vartheta): 0 \le \delta \le 1, 0 \le \vartheta \le 1\}$$
$$= \max\left\{\mathcal{H}(\varepsilon_0), \frac{16(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} + \frac{4(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^k[4]_q^k}\right\} \tag{27}$$

Thus, from Eqs. (24-26) and Eq. (27), the proof is completed.

Remark 3.5. For $\lambda = 0$ (and $\lambda = 1$) in Theorem 3.4, we can confirm the Hankel inequalities for the function classes $\mathcal{R}\Sigma_q^k(\phi)$, $\mathcal{H}\Sigma_q^k(\phi)$, respectively.

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