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Hankel Determinant for a Subclass of Bi-Univalent Functions Defined by Using a Symmetric q-Derivative Operator

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Abstract. In this paper, we discuss the various properties of a newly-constructed subclass of the class of normalized bi-univalent functions in the open unit disk, which is defined here by using a symmetric basic (or q-) derivative operator. Moreover, for functions belonging to this new basic (or q-) class of normalized bi-univalent functions, we investigate the estimates and inequalities involving the second Hankel determinant.

1. Introduction, Definitions and Notations

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions the form (1) which are also univalent in $\mathbb{U}$.

The *Koebe One-Quarter Theorem* [13] states that the image of $\mathbb{U}$ under every function f in the normalized univalent function class $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, clearly, every such univalent function has an inverse f^{-1} which satisfies the following condition:

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

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and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. For a brief history and interesting examples of functions in the class Σ, see the pioneering work on this subject by Srivastava *et al.* [46], which has apparently revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [46], we choose to recall the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familier Koebe function is not a member of the bi-univalent function class Σ. Such other common examples of functions in S as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ (see [46]).

Historically, Lewin [23] studied the class of bi-univalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [8] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \Sigma$. Later on, Netanyahu [28] showed that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [9] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^\star(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β $(0 \leqq \beta < 1)$ in $\mathbb{U}$, respectively (see [28]). The classes $\mathcal{S}^\star_\Sigma(\beta)$ and $\mathcal{K}_\Sigma(\beta)$ of bi-starlike functions of order β in $\mathbb{U}$ and bi-convex functions of order β in $\mathbb{U}$, corresponding to the function classes $\mathcal{S}^\star(\beta)$ and $\mathcal{K}(\beta)$, were also introduced analogously. For each of the function classes $\mathcal{S}^\star_\Sigma(\beta)$ and $\mathcal{K}_\Sigma(\beta)$, they found non-sharp estimates for the initial coefficients. Recently, motivated substantially by the aforementioned pioneering work on this subject by Srivastava *et al.* [46], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see, for example, [4], [15], [24], [40], [41], [42], [43], [44], [49] and [50]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geqq 4$. In the literature, there are only a few works determining the general coefficient bounds for $|a_n|$ for the analytic bi-univalent functions (see, for example, [2], [5], [11], [18], [19], [22] and [47]). The coefficient estimate problem for each of the coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\};\ \mathbb{N} = \{1, 2, 3, \cdots\})$ is still an open problem.

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for normalized univalent functions of the form given by (1) is well known for its rich history in Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [14] of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see, for details, [14]). The Fekete-Szegö functional has $\left|a_3 - \mu a_2^2\right|$ since received great attention, particularly in connection with many subclasses of the class $\mathcal{S}$ of normalized analytic and univalent functions (see, for example, [3], [25], [31], [45], [48] and [51]).

In the year 1976, Noonan and Thomas [29] defined the $\mathfrak{q}$th Hankel determinant of the function f in (1) by

$$H_{\mathfrak{q}}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\mathfrak{q}-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\mathfrak{q}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+\mathfrak{q}-1} & a_{n+\mathfrak{q}} & \cdots & a_{n+2\mathfrak{q}-2} \end{vmatrix} \qquad (n, \mathfrak{q} \in \mathbb{N};\ a_1 := 1).$$

The determinant $H_q(n)$ has also been considered by several other authors. For example, Noor [30] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained in the recent works [30] and [20] for different classes of functions.

We note, in particular, that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the well-known Fekete-Szegö functional. Very recently, the upper bounds of $H_2(2)$ for some specific analytic function classes were discussed by Deniz *et al.* [12] (see also [32]).

In the field of Geometric Function Theory, various subclasses of the normalized analytic function class $\mathcal{A}$ have been studied from different viewpoints. The q-calculus as well as the *fractional* q-calculus provide important tools that have been used in order to investigate various subclasses of $\mathcal{A}$. Historically speaking, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [39, pp. 347 *et seq.*]). In fact, the theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were used to construct several subclasses of analytic functions (see, for example, [1], [6], [26], [27], [33], [35], [36], [37] and [38]). In particular, Purohit and Raina [36] investigated applications of fractional q-calculus operators to define several classes of functions which are analytic in the open unit disk $\mathbb{U}$. On the other hand, Mohammed and Darus [26] studied approximation and geometric properties of these q-operators in regard to some subclasses of analytic functions in a compact disk.

We begin by providing some basic definitions and concept details of the q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall follow the notation and terminology in [39] and [16]. We first recall the definitions of fractional q-calculus operators of a complex-valued function $f(z)$.

Definition 1. Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^\lambda}{1 - q} & (\lambda \in \mathbb{C}) \\[2ex] \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 2. Let $q \in (0, 1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\[2ex] \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 3. For $q \in (0,1)$, $\lambda, \mu \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the q-shifted factorial $(\lambda; q)_\mu$ is defined by

$$(\lambda; q)_\mu := \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu + j}} \right) \qquad (\lambda, \mu \in \mathbb{C}),$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} \left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \end{cases}$$

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} \left(1 - \lambda\, q^j\right) \qquad (\lambda \in \mathbb{C}).$$

Definition 4. (see [21]; see also [39] and [16]) The q-derivative $D_q f$ of a function f is defined in a given subset of $\mathbb{C}$ by

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases} \tag{3}$$

provided that $f'(0)$ exists.

We note from Definition 4 that

$$\lim_{q \to 1-} \left(D_q f\right)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1) and (3) that

$$\left(D_q f\right)(z) = 1 + \sum_{n=2}^{\infty} [n]_q\, a_n z^{n-1}. \tag{4}$$

Definition 5. (see [7]) The symmetric q-derivative $\widetilde{D}_q f$ of a function f is defined in a given subset of $\mathbb{C}$ is by

$$\left(\widetilde{D}_q f\right)(z) = \begin{cases} \dfrac{f(qz) - f\left(q^{-1}z\right)}{\left(q - q^{-1}\right)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases} \tag{5}$$

provided that $f'(0)$ exists.

It easily follows from (1) and (5) that

$$\left(\widetilde{D}_q f\right)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1}, \tag{6}$$

where $\widetilde{[n]}_q$ denotes the number given by

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad (n \in \mathbb{N}),$$

which occurs frequently in the study of q-deformed quantum mechanical simple harmonic oscillator (see, for details, [10]).

The following properties hold true:

$$\widetilde{D}_q\Big(f(z) + g(z)\Big) = \Big(\widetilde{D}_q f\Big)(z) + \Big(\widetilde{D}_q g\Big)(z),$$

$$\begin{aligned}\widetilde{D}_q\Big(f(z)g(z)\Big) &= g\Big(q^{-1}z\Big)\Big(\widetilde{D}_q f\Big)(z) + f(qz)\Big(\widetilde{D}_q g\Big)(z)\\ &= g(qz)\Big(\widetilde{D}_q f\Big)(z) + f\Big(q^{-1}z\Big)\Big(\widetilde{D}_q g\Big)(z)\end{aligned}$$

and

$$\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}.$$

Finally, by comparing Definitions 4 and 5, we have the following relation:

$$\Big(\widetilde{D}_q f\Big)(z) = \Big(D_{q^2} f\Big)(q^{-1}z).$$

Moreover, by using (2) and (5), we also deduce that

$$\begin{aligned}\Big(\widetilde{D}_q g\Big)(w) &= \frac{g(qw) - g\Big(q^{-1}w\Big)}{(q - q^{-1})w}\\ &= 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q \left(2a_2^2 - a_3\right) w^2\\ &\qquad - \widetilde{[4]}_q \left(5a_2^3 - 5a_2a_3 + a_4\right) w^3 + \cdots . \end{aligned} \tag{7}$$

Definition 6. A function $f \in \Sigma$ is said to be in the class $\mathcal{H}_\Sigma(q;\beta)$ if the following conditions hold true:

$$\Re\left(\Big(\widetilde{D}_q f\Big)(z)\right) > \beta \qquad (0 \leqq \beta < 1;\ z \in \mathbb{U})$$

and

$$\Re\left(\Big(\widetilde{D}_q g\Big)(w)\right) > \beta \qquad (0 \leqq \beta < 1;\ w \in \mathbb{U}),$$

where $g = f^{-1}$.

We note from Definition 6 that

$$\begin{aligned}\lim_{q\to 1-} \mathcal{H}_\Sigma(q;\beta) &= \left\{ f : f \in \Sigma \ \text{ and } \ \left\{ \begin{array}{l} \lim\limits_{q\to 1-} \Re\left(\Big(\widetilde{D}_q f\Big)(z)\right) > \beta \ \ (z \in \mathbb{U}) \\ \lim\limits_{q\to 1-} \Re\left(\Big(\widetilde{D}_q g\Big)(w)\right) > \beta \ \ (w \in \mathbb{U}) \end{array} \right\} \right.\\ &= \mathcal{H}_\Sigma(\beta),\end{aligned}$$

where $\mathcal{H}_\Sigma(\beta)$ is the class of bi-univalent defined and studied by Srivastava *et al.* [46].

In this paper, we derive the upper bound for the functional

$$H_2(2) = a_2a_4 - a_3^2$$

for a function f, given by (1), which belongs to the bi-univalent function class $\mathcal{H}_\Sigma(q;\beta)$ given by Definition 6.

2. A Set of Lemmas

Let $\mathcal{P}$ be the class of functions $p(z)$ with positive real part consisting of all analytic functions $\mathcal{P} : \mathbb{U} \to \mathbb{C}$ satisfying the following conditions:

$$p(0) = 1 \qquad \text{and} \qquad \Re\Big(p(z)\Big) > 0.$$

Lemma 1. (see [34]) *If the function $p \in \mathcal{P}$ is defined by*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,$$

then

$$\left|p_n\right| \leqq 2 \qquad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

Lemma 2. (see [17]) *If the function $p \in \mathcal{P}$ is defined by*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,$$

then

$$2p_2 = p_1^2 + \xi\left(4 - p_1^2\right)$$

and

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1\xi - p_1\left(4 - p_1^2\right)\xi^2 + 2\left(4 - p_1^2\right)\left(1 - |\xi|^2\right)z$$

for some ξ and z with $|\xi| \leqq 1$ and $|z| \leqq 1$.

3. The Main Result and Its Consequences

Our main result in this paper is stated as the following theorem.

Theorem. *Let the function f given by* (1) *be in the class $\mathcal{H}_\Sigma(q;\beta)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq \begin{cases} 4(1-\beta)^2\left(\dfrac{4(1-\beta)^2}{\widetilde{[2]}_q^{\,4}} + \dfrac{1}{\widetilde{[2]}_q\widetilde{[4]}_q}\right) \\[2mm] \qquad \left(0 \leqq \beta \leqq 1 - \dfrac{\widetilde{[2]}_q^{\,2}\widetilde{[4]}_q + \widetilde{[2]}_q\sqrt{\widetilde{[2]}_q^{\,2}\widetilde{[4]}_q^{\,2} + 32_q\widetilde{[2]}_q\widetilde{[3]}_q^{\,2}\widetilde{[4]}_q}}{16\widetilde{[3]}_q\widetilde{[4]}_q}\right) \\[4mm] \dfrac{4(1-\beta)^2}{\widetilde{[3]}_q^{\,2}} - \dfrac{\left(\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta) + 3\widetilde{[2]}_q\widetilde{[3]}_q^{\,2} - 2\widetilde{[2]}_q^{\,2}\widetilde{[4]}_q\right)^2(1-\beta)^2}{\widetilde{[3]}_q^{\,2}\widetilde{[4]}_q\left(4\widetilde{[3]}_q^{\,2}\widetilde{[4]}_q(1-\beta)^2 - \widetilde{[2]}_q^{\,2}\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta) - 2\widetilde{[2]}_q^{\,3}\widetilde{[3]}_q^{\,2} + \widetilde{[2]}_q^{\,4}\widetilde{[4]}_q\right)} \\[4mm] \qquad \left(1 - \dfrac{\widetilde{[2]}_q^{\,2}\widetilde{[4]}_q + \widetilde{[2]}_q\sqrt{\widetilde{[2]}_q^{\,2}\widetilde{[4]}_q^{\,2} + 32_q\widetilde{[2]}_q\widetilde{[3]}_q^{\,2}\widetilde{[4]}_q}}{16\widetilde{[3]}_q\widetilde{[4]}_q} < \beta < 1\right). \end{cases} \tag{8}$$

Proof. Suppose that $f \in \mathcal{H}_\Sigma(q;\beta)$. Then

$$\left(\widetilde{D}_q f\right)(z) = \beta + (1-\beta)\vartheta(z) \tag{9}$$

and

$$\left(\widetilde{D}_q g\right)(w) = \beta + (1-\beta)\varphi(w), \tag{10}$$

where $g = f^{-1}$ and the functions $\vartheta \in \mathcal{P}$ and $\varphi \in \mathcal{P}$ are given by

$$\vartheta(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

and

$$\varphi(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots ,$$

respectively.

It follows from (9) and (10), together with (6) and (7), that

$$\widetilde{[2]}_q a_2 = (1-\beta)\, c_1, \tag{11}$$

$$\widetilde{[3]}_q a_3 = (1-\beta)\, c_2, \tag{12}$$

$$\widetilde{[4]}_q a_4 = (1-\beta)\, c_3 \tag{13}$$

and

$$-\widetilde{[2]}_q a_2 = (1-\beta)\, d_1, \tag{14}$$

$$\widetilde{[3]}_q \left(2a_2^2 - a_3\right) = (1-\beta)\, d_2, \tag{15}$$

$$-\widetilde{[4]}_q \left(a_4 + 5a_2^3 - 5a_2 a_3\right) = (1-\beta) d_3. \tag{16}$$

From (11) and (14), we obtain

$$c_1 = -d_1 \tag{17}$$

and

$$a_2 = \frac{1-\beta}{\widetilde{[2]}_q}\, c_1. \tag{18}$$

Upon subtracting (12) from (15), we have

$$a_3 = \frac{(1-\beta)^2}{\widetilde{[2]}_q^2}\, c_1^2 + \frac{1-\beta}{2\widetilde{[3]}_q}\, (c_2 - d_2). \tag{19}$$

Also, if we subtract (13) from (16), we get

$$a_4 = \frac{5\,(1-\beta)^2}{4\widetilde{[2]}_q \widetilde{[3]}_q}\, c_1 (c_2 - d_2) + \frac{1-\beta}{2\widetilde{[4]}_q}\, (c_3 - d_3). \tag{20}$$

Thus, by applying (18), (19) and (20), we find that

$$\begin{aligned}\left|a_2 a_4 - a_3^2\right| = \bigg| &-\frac{(1-\beta)^4}{\widetilde{[2]}_q^4}\, c_1^4 + \frac{(1-\beta)^3}{4\widetilde{[2]}_q^2 \widetilde{[3]}_q}\, c_1^2 (c_2 - d_2) \\ &+ \frac{(1-\beta)^2}{2\widetilde{[2]}_q \widetilde{[4]}_q}\, c_1 (c_3 - d_3) - \frac{(1-\beta)^2}{4\widetilde{[3]}_q^2}\, (c_2 - d_2)^2 \bigg|.\end{aligned} \tag{21}$$

Next, according to Lemma 2 and (17), we have

$$\left.\begin{array}{l} 2c_2 = c_1^2 + \xi\left(4 - c_1^2\right) \\ \\ 2d_2 = d_1^2 + \eta\left(4 - d_1^2\right) \end{array}\right\} \Longrightarrow c_2 - d_2 = \left(\frac{4 - c_1^2}{2}\right)(\xi - \eta) \tag{22}$$

and

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1\xi - c_1\left(4 - c_1^2\right)\xi^2 + 2\left(4 - c_1^2\right)\left(1 - |\xi|^2\right)z, \tag{23}$$

$$4d_3 = d_1^3 + 2\left(4 - d_1^2\right)d_1\eta - d_1\left(4 - d_1^2\right)\eta^2 + 2\left(4 - d_1^2\right)\left(1 - |\eta|^2\right)w \tag{24}$$

and

$$\begin{aligned} c_3 - d_3 = \frac{c_1^3}{2} + \frac{c_1\left(4 - c_1^2\right)}{2}(\xi + \eta) - \frac{c_1\left(4 - c_1^2\right)}{4}\left(\xi^2 + \eta^2\right) \\ + \frac{4 - c_1^2}{2}\left[\left(1 - |\xi|^2\right)z - \left(1 - |\eta|^2\right)w\right]. \end{aligned} \tag{25}$$

Then, by using (22), (23), (24) and (25) in (21), we get

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Bigg| & -\frac{(1-\beta)^4}{\widetilde{[2]}_q^4}c_1^4 + \frac{(1-\beta)^3}{4\widetilde{[2]}_q^2\widetilde{[3]}_q}c_1^2\frac{\left(4 - c_1^2\right)}{2}(\xi - \eta) + \frac{(1-\beta)^2}{4\widetilde{[2]}_q\widetilde{[4]}_q}c_1^4 \\ & + \frac{(1-\beta)^2}{2\widetilde{[2]}_q\widetilde{[4]}_q}c_1^2\frac{\left(4 - c_1^2\right)}{2}(\xi + \eta) - \frac{(1-\beta)^2}{2\widetilde{[2]}_q\widetilde{[4]}_q}c_1^2\frac{(4 - c_1^2)}{4}(x^2 + y^2) \\ & + \frac{(1-\beta)^2}{2\widetilde{[2]}_q\widetilde{[4]}_q}c_1\frac{(4 - c_1^2)}{2}\left[\left(1 - |x|^2\right)z - \left(1 - |y|^2\right)w\right] \\ & - \frac{(1-\beta)^2}{4\widetilde{[3]}_q^2}\frac{(4 - c_1^2)^2}{4}(\xi - \eta)^2\Bigg|. \end{aligned} \tag{26}$$

Since $\vartheta \in \mathcal{P}$, we find by applying Lemma 1 that $|c_1| \leqq 2$. Therefore, by letting $|c_1| = c$, we may assume without any loss of generality that $0 \leqq c \leqq 2$. We thus find from (26) that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \leqq & \frac{(1-\beta)^4}{\widetilde{[2]}_q^4}c^4 + \frac{(1-\beta)^2}{4\widetilde{[2]}_q\widetilde{[4]}_q}c^4 + \frac{(1-\beta)^2}{2\widetilde{[2]}_q\widetilde{[4]}_q}c(4 - c^2) \\ & + \left(\frac{(1-\beta)^3}{4\widetilde{[2]}_q^2\widetilde{[3]}_q}c^2\frac{\left(4 - c^2\right)}{2} + \frac{(1-\beta)^2}{2\widetilde{[2]}_q\widetilde{[4]}_q}c^2\frac{\left(4 - c^2\right)}{2}\right)(|\xi| + |\eta|) \\ & + \left(\frac{(1-\beta)^2}{2\widetilde{[2]}_q\widetilde{[4]}_q}c^2\frac{(4 - c^2)}{4} - \frac{(1-\beta)^2}{2\widetilde{[2]}_q\widetilde{[4]}_q}c\frac{(4 - c^2)}{2}\right)(|\xi|^2 + |\eta|^2) \\ & + \frac{(1-\beta)^2}{4\widetilde{[3]}_q^2}\frac{(4 - c^2)^2}{4}(|\xi| + |\eta|)^2. \end{aligned} \tag{27}$$

Now, for $\kappa = |\xi| \leqq 1$ and $\mu = |y| \leqq 1$, we can rewrite (27) in the following form:

$$\left|a_2a_4 - a_3^2\right| \leqq T_1 + (\kappa+\mu)\, T_2 + \left(\kappa^2+\mu^2\right) T_3 + (\kappa+\mu)^2\, T_4 =: G(\kappa,\mu) \tag{28}$$

where

$$T_1 = T_1(c) := \frac{(1-\beta)^2}{4}\left[\left(\frac{4\,(1-\beta)^2}{\widetilde{[2]}_q^4} + \frac{1}{\widetilde{[2]}_q\widetilde{[4]}_q}\right)c^4 + \frac{8c-2c^3}{\widetilde{[2]}_q\widetilde{[4]}_q}\right] \geqq 0,$$

$$T_2 = T_2(c) := \frac{(1-\beta)^2}{8}c^2(4-c^2)\left[\frac{(1-\beta)}{\widetilde{[2]}_q^2\widetilde{[3]}_q} + \frac{2}{\widetilde{[2]}_q\widetilde{[4]}_q}\right] \geqq 0,$$

$$T_3 = T_3(c) := \frac{(1-\beta)^2}{8\widetilde{[2]}_q\widetilde{[4]}_q}c(4-c^2)(c-2) \leqq 0$$

and

$$T_4 = T_4(c) := \frac{(1-\beta)^2}{16\widetilde{[3]}_q^2}(4-c^2)^2 \geqq 0.$$

We next need to maximize the function $G(\kappa,\mu)$ in (28) on the closed square $[0,1]\times[0,1]$. We must investigate the maximum value of $G(\kappa,\mu)$ according to $c \in (0,2)$, $c = 0$ and $c = 2$ by taking into account the sign of the following expression:

$$\mathfrak{G}_{\kappa,\mu} := G(\kappa,\kappa)\cdot G(\mu,\mu) - [G(\kappa,\mu)]^2.$$

Firstly, we let $c \in (0,2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $c \in (0,2)$, we conclude that

$$\mathfrak{G}_{\kappa,\mu} := G(\kappa,\kappa)\cdot G(\mu,\mu) - [G(\kappa,\mu)]^2 < 0.$$

Thus the function $G(\kappa,\mu)$ cannot have a local maximum in the interior of the square $[0,1]\times[0,1]$.

We now investigate the maximum value of the function $G(\kappa,\mu)$ on the boundary of the square $[0,1]\times[0,1]$. Indeed, for $\kappa = 0$ and $0 \leqq \mu \leqq 1$ (and, similarly, for $\mu = 0$ and $0 \leqq \kappa \leqq 1$), we obtain

$$G(0,\mu) =: H(\mu) = (T_3+T_4)\mu^2 + T_2\mu + T_1.$$

(i) The case when $T_3 + T_4 \geqq 0$: In this case, for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$, it is clear that

$$H'(\mu) = 2(T_3+T_4)\mu + T_2 > 0,$$

that is, that $H(\mu)$ is an increasing function. Hence, for fixed $c \in (0,2)$, the maximum value of $H(\mu)$ occurs at $\mu = 1$ and

$$\max\{H(\mu)\} = H(1) = T_1 + T_2 + T_3 + T_4.$$

(ii) The case when $T_3 + T_4 < 0$: Since $T_2 + 2(T_3+T_4) \geqq 0$ for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$, it is clear that

$$T_2 + 2(T_3+T_4) < 2(T_3+T_4)\mu + T_2 < T_2$$

and so $H'(\mu) > 0$. Hence, for fixed $c \in (0,2)$, the maximum value of $H(\mu)$ occurs at $\mu = 1$.

For $\kappa = 1$ and $0 \leqq \mu \leqq 1$ (and, similarly, for $\mu = 1$ and $0 \leqq \kappa \leqq 1$), we obtain

$$G(1,\mu) = F(\mu) = (T_3+T_4)\mu^2 + (T_2+2T_4)\mu + T_1 + T_2 + T_3 + T_4.$$

Analogous to the above cases of $T_3 + T_4$, we find that

$$\max\{F(\mu)\} = F(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $H(1) \leqq F(1)$ for $c \in (0, 2)$,

$$\max\{G(\kappa, \mu)\} = G(1, 1)$$

on the boundary of the square $[0, 1] \times [0, 1]$. Thus the maximum value of the function $G(\kappa, \mu)$ occurs at $\kappa = 1$ and $\mu = 1$ in the closed square $[0, 1] \times [0, 1]$.

Let $K : (0, 2) \to \mathbb{R}$

$$K(c) := \max\{G(\kappa, \mu)\} = G(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{29}$$

Upon substituting the values of T_1, T_2, T_3 and T_4 into the function $K(c)$ defined by (29), we obtain

$$\begin{aligned} K(c) = (1-\beta)^2 \Bigg[& \left(\frac{(1-\beta)^2}{\widetilde{[2]}_q^4} - \frac{(1-\beta)}{4\widetilde{[2]}_q^2 \widetilde{[3]}_q} - \frac{1}{2\widetilde{[2]}_q \widetilde{[4]}_q} + \frac{1}{4\widetilde{[3]}_q^2} \right) c^4 \\ & + \left(\frac{(1-\beta)}{\widetilde{[2]}_q^2 \widetilde{[3]}_q} + \frac{3}{\widetilde{[2]}_q \widetilde{[4]}_q} - \frac{2}{\widetilde{[3]}_q^2} \right) c^2 + \frac{4}{\widetilde{[3]}_q^2} \Bigg]. \end{aligned} \tag{30}$$

Let us assume that the function $K(c)$ has a maximum value at an interior point $c \in (0, 2)$. Then, by elementary calculation using (30), we have

$$\begin{aligned} K'(c) = (1-\beta)^2 \Bigg[& \left(\frac{4\,(1-\beta)^2}{\widetilde{[2]}_q^4} - \frac{(1-\beta)}{\widetilde{[2]}_q^2 \widetilde{[3]}_q} - \frac{2}{\widetilde{[2]}_q \widetilde{[4]}_q} + \frac{1}{\widetilde{[3]}_q^2} \right) c^3 \\ & + \left(\frac{2\,(1-\beta)}{\widetilde{[2]}_q^2 \widetilde{[3]}_q} + \frac{6}{\widetilde{[2]}_q \widetilde{[4]}_q} - \frac{4}{\widetilde{[3]}_q^2} \right) c \Bigg]. \end{aligned} \tag{31}$$

By means of some calculations, we can examine the following two cases.

Case 1. Suppose that

$$\frac{4\,(1-\beta)^2}{\widetilde{[2]}_q^4} - \frac{(1-\beta)}{\widetilde{[2]}_q^2 \widetilde{[3]}_q} - \frac{2}{\widetilde{[2]}_q \widetilde{[4]}_q} + \frac{1}{\widetilde{[3]}_q^2} \geqq 0.$$

Therefore, we have

$$0 \leqq \beta \leqq 1 - \frac{\widetilde{[2]}_q^2 \widetilde{[4]}_q + \widetilde{[2]}_q \sqrt{\widetilde{[2]}_q \widetilde{[4]}_q \left(32\widetilde{[3]}_q^2 - 15\widetilde{[2]}_q \widetilde{[4]}_q \right)}}{8\widetilde{[3]}_q \widetilde{[4]}_q}$$

and $K'(c) > 0$ for $c \in (0, 2)$. Since $K(c)$ is an increasing function in the interval $(0, 2)$, it has no maximum value in this interval.

Case 2. Suppose that

$$\frac{4\,(1-\beta)^2}{\widetilde{[2]}_q^4} - \frac{(1-\beta)}{\widetilde{[2]}_q^2 \widetilde{[3]}_q} - \frac{2}{\widetilde{[2]}_q \widetilde{[4]}_q} + \frac{1}{\widetilde{[3]}_q^2} < 0.$$

Therefore, we have

$$1-\frac{\widetilde{[2]}_q^2\widetilde{[4]}_q+\widetilde{[2]}_q\sqrt{\widetilde{[2]}_q\widetilde{[4]}_q\left(32\widetilde{[3]}_q^2-15\widetilde{[2]}_q\widetilde{[4]}_q\right)}}{8\widetilde{[3]}_q\widetilde{[4]}_q}<\beta<1.$$

In this case, $K'(c)=0$ implies the real critical point $c=c_0^{(1)}$, where

$$c_0^{(1)}=\sqrt{\frac{-2\widetilde{[2]}_q^2\left(\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta)+3\widetilde{[2]}_q\widetilde{[3]}_q^2-2\widetilde{[2]}_q^2\widetilde{[4]}_q\right)}{4\widetilde{[3]}_q^2\widetilde{[4]}_q(1-\beta)^2-\widetilde{[2]}_q^2\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta)-2\widetilde{[2]}_q^3\widetilde{[3]}_q^2+\widetilde{[2]}_q^4\widetilde{[4]}_q}}.$$

For the parameter β constrained by

$$\begin{aligned}1-\frac{\widetilde{[2]}_q^2\widetilde{[4]}_q+\widetilde{[2]}_q\sqrt{\widetilde{[2]}_q\widetilde{[4]}_q\left(32\widetilde{[3]}_q^2-15\widetilde{[2]}_q\widetilde{[4]}_q\right)}}{8\widetilde{[3]}_q\widetilde{[4]}_q}&<\beta\\&\leqq 1-\frac{\widetilde{[2]}_q^2\widetilde{[4]}_q+\widetilde{[2]}_q\sqrt{\widetilde{[2]}_q^2\widetilde{[4]}_q^2+32\widetilde{[2]}_q\widetilde{[3]}_q^2\widetilde{[4]}_q}}{16\widetilde{[3]}_q\widetilde{[4]}_q},\end{aligned}$$

we observe that $c_0^{(1)}\geqq 2$, that is, that $c_0^{(1)}$ is outside of the interval $(0,2)$. On the other hand, when the parameter β is constrained by

$$1-\frac{\widetilde{[2]}_q^2\widetilde{[4]}_q+\widetilde{[2]}_q\sqrt{\widetilde{[2]}_q^2\widetilde{[4]}_q^2+32\widetilde{[2]}_q\widetilde{[3]}_q^2\widetilde{[4]}_q}}{16\widetilde{[3]}_q\widetilde{[4]}_q}<\beta<1,$$

we observe that $c_0(1)<2$, that is, that $c_0^{(2)}$ is an interior point of the closed interval $[0,2]$. Since $K''\left(c_0^{(2)}\right)<0$, the maximum value of $K(c)$ occurs at $c=c_0^{(2)}$. Thus, clearly, we have

$$\begin{aligned}&K\left(c_0^{(2)}\right)=(1-\beta)^2\\&\cdot\left(\frac{4}{\widetilde{[3]}_q^2}-\frac{\left(\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta)+3\widetilde{[2]}_q\widetilde{[3]}_q^2-2\widetilde{[2]}_q^2\widetilde{[4]}_q\right)^2}{\widetilde{[3]}_q^2\widetilde{[4]}_q\left(4\widetilde{[3]}_q^2\widetilde{[4]}_q(1-\beta)^2-\widetilde{[2]}_q^2\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta)-2\widetilde{[2]}_q^3\widetilde{[3]}_q^2+\widetilde{[2]}_q^4\widetilde{[4]}_q\right)}\right).\end{aligned}\tag{32}$$

Secondly, in the case when $c=2$, we obtain

$$G(\kappa,\mu)=4(1-\beta)^2\left(\frac{4(1-\beta)^2}{\widetilde{[2]}_q^4}+\frac{1}{\widetilde{[2]}_q\widetilde{[4]}_q}\right).\tag{33}$$

Finally, in the case when $c=0$, we find that

$$G(\kappa,\mu)=\frac{(1-\beta)^2}{\widetilde{[3]}_q^2}(\kappa+\mu)^2.$$

We can easily see that the maximum value of the function $G(\kappa, \mu)$ occurs at $\kappa = \mu = 1$:

$$\max\{G(\kappa, \mu)\} = G(1, 1) = \frac{4(1-\beta)^2}{\widetilde{[3]}_q^2}. \tag{34}$$

We thus find from (32), (33) and (34) that

$$\begin{aligned}\frac{4(1-\beta)^2}{\widetilde{[3]}_q^2} < \frac{16(1-\beta)^4}{\widetilde{[2]}_q^4} + \frac{4(1-\beta)^2}{\widetilde{[2]}_q\widetilde{[4]}_q} < \frac{4(1-\beta)^2}{\widetilde{[3]}_q^2} \\ - \frac{\left(\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta) + 3\widetilde{[2]}_q\widetilde{[3]}_q^2 - 2\widetilde{[2]}_q^2\widetilde{[4]}_q\right)^2 (1-\beta)^2}{\widetilde{[3]}_q^2\widetilde{[4]}_q\left(4\widetilde{[3]}_q^2\widetilde{[4]}_q(1-\beta)^2 - \widetilde{[2]}_q^2\widetilde{[3]}_q\widetilde{[4]}_q(1-\beta) - 2\widetilde{[2]}_q^3\widetilde{[3]}_q^2 + \widetilde{[2]}_q^4\widetilde{[4]}_q\right)}\end{aligned} \tag{35}$$

for the parameter β constrained by

$$1 - \frac{\widetilde{[2]}_q^2\widetilde{[4]}_q + \widetilde{[2]}_q\sqrt{\widetilde{[2]}_q^2\widetilde{[4]}_q^2 + 32_q\widetilde{[2]}_q\widetilde{[3]}_q^2\widetilde{[4]}_q}}{16\widetilde{[3]}_q\widetilde{[4]}_q} < \beta < 1.$$

This leads us to the second inequality of (8).

On the other hand, from the left-hand part of the inequality (35), we get

$$\frac{4(1-\beta)^2}{\widetilde{[3]}_q^2} < \frac{16(1-\beta)^4}{\widetilde{[2]}_q^4} + \frac{4(1-\beta)^2}{\widetilde{[2]}_q\widetilde{[4]}_q},$$

which yields the first inequality of (8) for the parameter β constrained by

$$0 \leqq \beta \leqq 1 - \frac{\widetilde{[2]}_q^2\widetilde{[4]}_q + \widetilde{[2]}_q\sqrt{\widetilde{[2]}_q^2\widetilde{[4]}_q^2 + 32_q\widetilde{[2]}_q\widetilde{[3]}_q^2\widetilde{[4]}_q}}{16\widetilde{[3]}_q\widetilde{[4]}_q}.$$

This evidently completes the proof of the above Theorem. □

Corollary. *Let the function f given by* (1) *be in the class $H_\Sigma(\beta)$ $(0 \leqq \beta < 1)$. Then*

$$\begin{aligned}&\left|a_2a_4 - a_3^2\right| \\ &\leqq \begin{cases} (1-\beta)^2\left((1-\beta)^2 + \frac{1}{2}\right) & \left(0 \leqq \beta \leqq \frac{11-\sqrt{37}}{12}\right) \\ \left(\frac{1-\beta}{3}\right)^2\left(4 - \frac{(17-6\beta)^2}{16(9\beta^2-15\beta+1)}\right) & \left(\frac{11-\sqrt{37}}{12} < \beta < 1\right). \end{cases}\end{aligned} \tag{36}$$

4. Concluding Remarks and Observations

In our present investigation, we have derived the various properties of a newly-constructed subclass $\mathcal{H}_\Sigma(q;\beta)$ $(0 < q < 1;\ 0 \leqq \beta < 1)$ of the class Σ of normalized bi-univalent functions in the open unit disk $\mathbb{U}$. We have defined this two-parameter function class $\mathcal{H}_\Sigma(q;\beta)$ by making use of a symmetric basic (or q-) derivative operator. For functions belonging to this bi-univalent function class, we have found the estimates and inequalities for the second Hankel determinant. The corresponding result is also derived for the function class $H_\Sigma(\beta)$ $(0 \leqq \beta < 1)$ which was introduced and studied earlier by Srivastava *et al.* [46].

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