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CERTAIN CLASSES OF THE MEROMORPHIC HARMONIC FUNCTIONS WITH A POLE AT SOME FIXED POINT OF THE UNIT DISK

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Abstract. The class $S_H(p)$, $0 \leq p < 1$, of complex valued, meromorphic harmonic univalent sense-preserving functions in the unit disk $U\backslash\{p\}$ is studied. The functions belong to $S_H(p)$ have the expansion $f(z) = \frac{\alpha}{z-p} + \sum_{n=0}^{\infty} c_n z^n + \overline{\sum_{n=1}^{\infty} d_n z^n} + A \log|z-p|$ and $\lim_{z\to p} f(z) = \infty$. Some coefficient estimates, distortion and area theorems are obtained. Sufficient coefficient conditions for a class of meromorphic harmonic univalent sense-preserving functions that are starlike and convex are given.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]). There are numerous papers on univalent harmonic functions defined on the domain $U = \{z : |z| < 1\}$ (see [1, 7, 10] and [8]). Hengartner and Schober [4], investigated functions harmonic in the exterior of the unit disk $\tilde{U} = \{z : |z| > 1\}$, among other things they showed that complex value, harmonic, orientation preserving univalent mapping f, defined in $\tilde{U}$ and satisfying $f(\infty) = \infty$, must admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log|z|$$

where

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$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n} \qquad \text{and} \qquad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}$$

$0 \leq |\beta| < |\alpha|$, and $a(z) = \overline{f_{\bar{z}}}/f_z$ is analytic and satisfies $|a(z)| < 1$ for $z \in \tilde{U}$. Recently, Jahangiri [5] , Jahangiri and Silverman [6] and Murugusundaramoorthy [9] focused the study to the family of harmonic meromorphic functions.

For $0 \leq p < 1$, we let $S_H(p)$ denote the class of functions harmonic univalent, sense-preserving and meromorphic in U, with $\lim_{z \to p} f(z) = \infty$ and which are the representation

$$f(z) = h(z) + \overline{g(z)} + A \log|z - p| \tag{1}$$

where

$$h(z) = \frac{\alpha}{z - p} + \sum_{n=0}^{\infty} c_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} d_n z^n, \qquad \alpha \in \mathbb{C}. \tag{2}$$

By applying an affine post-mapping to f we may normalize f so that $c_0 = 0$ in the representation (2). We further remove the logarithmic singularity by letting $A = 0$ and so focus our attention to the subclass $S'_H(p)$ of all harmonic, sense-preserving, univalent, meromorphic mappings which have the development

$$f(z) = h(z) + \overline{g(z)} \tag{3}$$

where

$$h(z) = \frac{\alpha}{z - p} + \sum_{n=1}^{\infty} c_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} d_n z^n, \quad z \in U \backslash \{p\}, \tag{4}$$

or we may set for $z \in U_p = \{z : 0 < |z - p| < 1 - p\}$

$$h(z) = \frac{\alpha}{z - p} + \sum_{n=1}^{\infty} a_n (z - p)^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n (z - p)^n. \tag{5}$$

In this paper, we give some coefficient estimates, area theorem and distortion theorem for function in $S_H(p)$ and its subclass $S'_H(p)$. Also, sufficient coefficient conditions for a class of meromorphic harmonic sense-preserving functions that are starlike and convex are given.

2. Main Results

Theorem 2.1. *Let h and g has expansion* (4). *If*

(6) $$\sum_{n=1}^{\infty} n(|c_n|+|d_n|) \leq \frac{|\alpha|}{(1+p)^2}$$

then $f = h+\overline{g}$ *is harmonic univalent, sense preserving in* $U\backslash\{p\}$ *and* $f \in S'_H(p)$, $0 \leq p < 1$. *Also, for* $|z| = r < 1$

$$\frac{|\alpha|\,(1+p-r)}{(1+p)^2} < |f(z)| < \frac{|\alpha|\,(1+p+r)}{|r-p|(1+p)}.$$

Proof. For $z_1 \neq p$, $z_2 \neq p$, and $|z_1| \leq |z_2| < 1$ we have

$$\begin{aligned}
&|f(z_1)-f(z_2)| \\
&\geq |h(z_1)-h(z_2)| - |g(z_1)-g(z_2)| \\
&\geq \frac{|z_1-z_2|}{|z_1-p|\,|z_2-p|}\left[|\alpha| - |z_1-p|\,|z_2-p| \sum_{n=1}^{\infty} n(|c_n|+|d_n|)\,|z_2|^{n-1}\right] \\
&> \frac{|z_1-z_2|}{|z_1-p|\,|z_2-p|}\left[|\alpha| - (1+p)^2 \sum_{n=1}^{\infty} n(|c_n|+|d_n|)\,|z_2|\right] \\
&> \frac{|z_1-z_2|}{|z_1-p|\,|z_2-p|}\left[|\alpha| - (1+p)^2 \sum_{n=1}^{\infty} n(|c_n|+|d_n|)\right] \geq 0, \text{ by (6)},
\end{aligned}$$

and f is univalent in $U\backslash\{p\}$. To show that f is sense preserving in $U\backslash\{p\}$, we need to show that $|h'(z)| > |g'(z)|$ in $U\backslash\{p\}$. We have

$$\begin{aligned}
\left|h'(z)\right| &= \frac{1}{|z-p|^2}\left|\alpha - (z-p)^2 \sum_{n=1}^{\infty} n c_n z^{n-1}\right| \\
&\geq \frac{1}{|z-p|^2}\left[|\alpha| - |z-p|^2 \sum_{n=1}^{\infty} n|c_n|\right] \\
&> \frac{1}{(1+p)^2}\left[|\alpha| - (1+p)^2 \sum_{n=1}^{\infty} n|c_n|\right] \\
&\geq \sum_{n=1}^{\infty} n|c_n| > \sum_{n=1}^{\infty} n|d_n||z|^{n-1} \\
&\geq \left|g'(z)\right|.
\end{aligned}$$

For $|z| = r < 1$, we see from (3) and (4)

$$|f(z)| = \left|\frac{\alpha}{z-p} + \sum_{n=1}^{\infty} c_n z^n + \overline{\sum_{n=1}^{\infty} d_n z^n}\right|$$

$$\geq \frac{1}{|z-p|}\left[|\alpha| - |z-p|\sum_{n=1}^{\infty}(|c_n|+|d_n|)|z|^n\right]$$
$$> \frac{1}{(1+p)}\left[|\alpha| - (1+p)\sum_{n=1}^{\infty}(|c_n|+|d_n|)r\right]$$
$$\geq \frac{|\alpha|\,(1+p-r)}{(1+p)^2}, \quad \text{by (6)},$$

and

$$|f(z)| = \left|\frac{\alpha}{z-p} + \sum_{n=1}^{\infty} c_n z^n + \overline{\sum_{n=1}^{\infty} d_n z^n}\right|$$
$$\leq \frac{1}{|z-p|}\left[|\alpha| + |z-p|\sum_{n=1}^{\infty}(|c_n|+|d_n|)|z|^n\right]$$
$$< \frac{1}{|r-p|}\left[|\alpha| + (1+p)\sum_{n=1}^{\infty}(|c_n|+|d_n|)r\right]$$
$$\leq \frac{|\alpha|\,(1+p+r)}{|r-p|\,(1+p)}, \quad \text{by (6)}. \qquad \blacksquare$$

By Theorem 2.1 the family of $f \in S'_H(p)$, which satisfies the condition (6) is locally uniformly bounded family of harmonic function, hence in question is normal.

Theorem 2.2. (a) *If* $f \in S_H(p)$, *then* $|A| \leq \frac{2|\alpha|}{1-p}$ *and* $|b_1| \leq \frac{|\alpha|}{1-p}$.

(b) *If* $f \in S'_H(p)$, *then* $|b_1| \leq \dfrac{|\alpha|}{(1-p)^2}$ *and* $|b_2| \leq \frac{|\alpha|}{2(1-p)^3}$.

Proof. If $f \in S_H(p)$ has expansion (5), then sense-preserving property of f implies that the Jacobian $|f_z|^2 - |f_{\bar{z}}|^2$ is positive, and so

$$|f_{\bar{z}}(z)| = |g'(z) + \frac{\bar{A}}{2(z-p)}| \leq |f_z| = \left|h'(z) + \frac{A}{2(z-p)}\right|.$$

If the latter were to vanish identically, then f would be constant and not univalent. Therefore

$$a(z) = \frac{\overline{f_{\bar{z}}}(z)}{f_z(z)} = \frac{2(z-p)^2 g'(z) + \bar{A}(z-p)}{2(z-p)^2 h'(z) + A(z-p)}$$
$$= \frac{2(z-p)^2 g'(z) + \bar{A}(z-p)}{-2\alpha + 2(z-p)^2\left(1 + \sum_{n=1}^{\infty} n a_n z^{n-1}\right) + A(z-p)}$$

is analytic in U and $|a(z)| < 1$. We shall use the bounds $|w_0| \leq 1$ and $|w_1| \leq 1 - |w_0|^2$ for analytic functions $w(z) = w_0 + w_1 z^{-1} + ...$ in $\tilde{U}$ that bounded by one. Let

$$\varphi(z) = \frac{1-p}{z} + p.$$

be conformal mapping from $\tilde{U}$ to U_p. The composite functions

$$\begin{aligned} k(z) \;=\; a(\varphi(z)) &= \frac{2(\varphi(z)-p)^2 g'(\varphi(z)) + \bar{A}(\varphi(z)-p)}{-2\alpha + 2(\varphi(z)-p)^2\left(1+\sum_{n=2}^{\infty} n a_n z^{n-1}\right) + A(\varphi(z)-p)} \\ &= -(1-p)\frac{\bar{A}}{2\alpha}z^{-1} - (1-p)^2\left(\frac{b_1}{\alpha} + \frac{|A|^2}{4\alpha^2}\right)z^{-2} \\ &\quad -(1-p)^3\left(\frac{2b_2}{\alpha} + \frac{\bar{A}a_1}{2\alpha^2} + \frac{A}{2\alpha^2} + \frac{A|A|^2}{8\alpha^3}\right)z^{-3} + ... \end{aligned}$$

is analytic function in $\tilde{U}$ and $|k(z)| < 1$ by sense-preserving property of $f(z)$. The maximum principle implies that $w(z) = zk(z)$ is also bounded by one, and so

$$\left|\frac{(1-p)\bar{A}}{2\alpha}\right| \leq 1 \tag{7}$$

and

$$\left|\frac{(1-p)^2 b_1}{\alpha} + \frac{(1-p)^2}{4\alpha^2}|A|^2\right| \leq 1 - \left|\frac{(1-p)\bar{A}}{2\alpha}\right|^2.$$

The latter implies

$$|b_1| \leq \frac{|\alpha|}{1-p}.$$

If $f \in S'_H(p)$, then $A = 0$,

$$k(z) = -\frac{(1-p)^2 b_1}{\alpha}z^{-2} - \frac{2(1-p)^3 b_2}{\alpha}z^{-3} + ...$$

and $w(z) = z^2 k(z)$ is bounded by one. Therefore

$$\left|\frac{(1-p)^2}{\alpha}b_1\right| \leq 1$$

and

$$\left|\frac{2(1-p)^3 b_2}{\alpha}\right| \leq 1 - \left|\frac{(1-p)^2 b_1}{\alpha}\right|^2$$

or

$$|b_2| \leq \frac{|\alpha|}{2(1-p)^3}\left(1 - \left|\frac{(1-p)^2 b_1}{\alpha}\right|^2\right) \leq \frac{|\alpha|}{2(1-p)^3}. \qquad \blacksquare$$

The coefficient bounds in Theorem 2.3 are all sharp. Equality in (a) is attained, for example, by the function

$$f(z) = \frac{\alpha}{z-p} - \frac{\alpha p}{1-p} + \frac{\alpha}{1-p}\bar{z} + \frac{2\alpha}{1-p}\log|z-p|.$$

In (b) the bound for b_1 is sharp for the function

$$f(z) = \frac{\alpha}{z-p} + \frac{\alpha}{(1-p)^2}\bar{z}$$

W. Hengartner and G. Schober [4] proved the following lemma which we shall use the next theorem. This lemma contains a distortion estimate for a class locally quasi conform mapping.

Lemma 2.3. *Let f be a diffeomorphism of U satisfying*

$$|f_{\bar{z}}(z)| \leq |z||f_z(z)| \quad \textit{forall } z \in U,$$

$$f(z) = z + O(|z|^{\beta}) \quad \textit{for some } \beta > 1 \textit{ as } z \to 0.$$

Then for all $z \in U$ we have

$$|f(z)| \geq \frac{|z|}{4(1+|z|)^2}.$$

In particular, the disk $\{w : |w| < \frac{1}{16}\}$ is contained in $f(U)$.

An immediate consequence is the following distortion theorem for the nonvanishing class

$$S_H^0(p) = \{f - c : f \in S_H^{'}(p) \text{ and } c \notin f(U_p)\}.$$

Theorem 2.4. *If $f(z) = \frac{\alpha}{z-p} - c + \sum_{n=1}^{\infty} a_n(z-p)^n + \overline{\sum_{n=1}^{\infty} b_n(z-p)^n}$ belongs to $S_H^0(p)$, then*

$$|f(z)| \leq \frac{4|\alpha|(1-p+|z-p|)^2}{(1-p)^2|z-p|}, \ z \in U_p,$$

$f(U_p)$ contains the set $\{w : |w| > \frac{16|\alpha|}{1-p}\}$, and $|c| \leq \frac{16|\alpha|}{1-p}$.

Proof. If f belongs to $S_H^0(p)$ and has expansion (5), then

$$\tilde{f}(z) = \frac{\alpha}{(1-p)f((1-p)z+p)}$$

is a diffeomorphism of U that satisfies

$$|\tilde{f}_{\bar{z}}(z)|/|\tilde{f}_z(z)| = |a((1-p)z+p)| \le |z|$$

$$\tilde{f}(z) = z + z^2 + O(|z|^2) \quad \text{as} \quad z \to 0.$$

Therefore Lemma 2.3 applies to $\tilde{f}$, and the first two conclusions follow.

$$|c| = \left| \frac{1}{2} \int_0^{2\pi} f(re^{i\theta}) d\theta \right| \le \frac{4|\alpha|(1-p+r)^2}{(1-p)^2 r}$$

for all $r < 1-p$. Let r approach $1-p$ to obtain $|c| \le \frac{16|\alpha|}{1-p}$. ■

The following theorem contains lower bound for the diameter of the omitted set $\mathbb{C}\backslash f(U_p)$ depending on the coefficient b_1 and α.

Theorem 2.5. *If h and g has expansion (5) and $f = h + \bar{g} \in S_H(p)$ then the diameter D_f of $\mathbb{C}\backslash f(U_p)$ satisfies*

$$D_f \ge \frac{2|\alpha + (1-p)^2 b_1|}{1-p}.$$

This estimate is sharp for

$$f(z) = \frac{\alpha}{z-p} + b_1\bar{z} - b_1 p + A \log |z-p|$$

whenever $|b_1| = |\alpha|/(1-p)^2$ *and* $A = 0$, $b_1 = -\alpha/(1-p)^2$ *and* $|A| \le 2|\alpha|/(1-p)$, *or* $|b_1| < |\alpha|/(1-p)^2$ *and* $|A| \le \left[|\alpha|^2 - (1-p)^4|b_1|^2\right] / \left[(1-p)^2 \left|\left(|\alpha| + (1-p)^2 b_1\right)/|\alpha|\right|\right]$.

Proof. Let $D_f(r)$ be diameter of $f(|z-p| = r)$, $0 < r < 1-p$, and let $D_f^*(r) = \max_{|z-p|=r} |f(z) - f(-z)|$. Then $D_f(r) \searrow D_f$ as $r \to 1-p$ and $D_f(r) \ge D_f^*(r)$. Since

$$\begin{aligned} D_f^*(r)^2 &\ge \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(-re^{i\theta})|^2 d\theta \\ &= 4 \left[\frac{|\alpha|^2}{r^2} + \bar{b}_1\alpha + |a_1|^2 r^2 + |b_1|^2 r^2 + \sum_{n=1}^{\infty} \left(|a_{2n+1}|^2 + |b_{2n+1}|^2\right) r^{2(2n+1)} \right] \\ &\ge 4|b_1 r + \frac{\alpha}{r}|^2, \end{aligned}$$

we conclude that $D_f \geq \frac{2|\alpha+(1-p)^2 b_1|}{1-p}$. ■

Theorem 2.6. *If h and g has expansion (4) and $f = h + \bar{g} \in S_H(p)$ then the diameter D_f of $\mathbb{C} \backslash f(\cup \backslash \{p\})$ satisfies*

$$D_f \geq 2\,|\alpha + d_1|\,.$$

This estimate is sharp for

$$f(z) = \frac{\alpha}{z-p} + b_1 \bar{z} - b_1 p + A \log|z-p|$$

whenever $|b_1| = |\alpha|/(1-p)^2$ and $A = 0$, $b_1 = -\alpha/(1-p)^2$ and $|A| \leq 2|\alpha|/(1-p)$, or $|b_1| < |\alpha|/(1-p)^2$ and $|A| \leq \left[|\alpha|^2 - (1-p)^4 |b_1|^2\right] / \left[(1-p)^2 \left|\left(|\alpha| + (1-p)^2\right.\right.\right.$ $\left.\left.\left.b_1\right) / |\alpha|\right|\right]$.

Proof. Let $D_f(r)$ be diameter of $f(|z| = r)$, $p < r < 1$, and let $D_f^*(r) = \max_{|z|=r} |f(z) - f(-z)|$. Then $D_f(r) \searrow D_f$ as $r \to 1$ and $D_f(r) \geq D_f^*(r)$. Since

$$\begin{aligned} D_f^*(r)^2 \;\geq\; & 4\left[\sum_{n=1}^{\infty} \frac{|\alpha|^2 p^{2(2n-2)}}{r^{2(2n-1)}} + 2\sum_{n=1}^{\infty} p^{2n-2}\,\mathrm{Re}(\alpha d_{2n-1}) + \sum_{n=1}^{\infty} |d_{2n-1}|^2 r^{2(2n-1)} \right. \\ & \left. + \sum_{n=1}^{\infty} |c_{2n-1}|^2 r^{2(2n-1)}\right] \geq \sum_{n=1}^{\infty} \left|\frac{\alpha p^{2n-2}}{r^{2n-1}} + d_{2n-1} r^{2n-1}\right|^2 \\ \geq\; & 4\left|\frac{\alpha}{r} + d_1 r\right|^2, \end{aligned}$$

we conclude that $D_f \geq 2\,|\alpha + d_1|$. ■

The next theorems is classical area theorems.

Theorem 2.7. *If $f = h + \bar{g} \in S'_H(p)$ has expansion (5) then*

$$\sum_{n=1}^{\infty} n(1-p)^{2n}(|a_n|^2 - |b_n|^2) \leq \frac{|\alpha|^2 + 2(1-p)^2\,\mathrm{Re}(\alpha b_1)}{(1-p)^2}.$$

Also, if $f = h + \bar{g}$ has expansion (4), then

$$\sum_{n=1}^{\infty} n(|c_n|^2 - |d_n|^2) - 2\alpha \sum_{n=1}^{\infty} n p^{n-1}\,\mathrm{Re}(d_n) \leq \frac{|\alpha|^2}{(1-p^2)^2}$$

Equality occurs if and only if $\mathbb{C} \backslash f(U_p)$ and $\mathbb{C} \backslash f(U \backslash \{p\})$ have area zero, respectively.

Proof. The area of the omitted set is

$$\lim_{r\to 1-p}\frac{1}{2i}\int\limits_{|z-p|=r}\bar{f}df = \left[\lim_{r\to 1}\frac{1}{2i}\int\limits_{|z-p|=r}\bar{h}h'dz+\frac{1}{2i}\int\limits_{|z-p|=r}g\overline{g'}d\bar{z}\right.$$
$$\left.+\frac{1}{2i}\int\limits_{|z-p|=r}gh'dz+\frac{1}{2i}\int\limits_{|z-p|=r}\bar{h}\overline{g'}d\bar{z}\right]$$
$$=\pi\left[\sum_{n=1}^{\infty}n(|a_n|^2-|b_n|^2)-\frac{|a|^2}{(1-p^2)^2}-2\alpha\sum_{n=1}^{\infty}np^{n-1}\operatorname{Re}(b_n)\right]$$

For $0<|z-p|=r<1-p$ the curve $\Gamma_r=f(C_r)$ is a simple closed curve oriented clockwise. Hence, for $r\to 1-p$ we obtain

$$\sum_{n=1}^{\infty}n(|a_n|^2-|b_n|^2)-\frac{|a|^2}{(1-p)^2}-2\operatorname{Re}(\alpha b_1)\le 0,$$

and the result follows. ■

Denote by $S_H^*(p)$ the nonvanishing subclass of $S'_H(p)$ consisting of functions f of the forms (3) and (5) that are map U_p onto the complement of a point-set starlike with respect to origin.

Theorem 2.8. *Let f be of the forms (3) and (5). If*

$$\sum_{n=1}^{\infty}n(|a_n|+|b_n|)\le\frac{|\alpha|}{(1+p)^2}, \tag{8}$$

then f is harmonic univalent, sense preserving in U_p and $f\in S_H^*(p)$.

Proof. By using the same method as Theorem 2.1, we obtain that f is harmonic univalent and sense preserving in U_p. Now, we need to show that f is in $S_H^*(p)$. A necessary and sufficient condition for such f to be starlike in U_p is that for each z, $0<|z-p|=r<1-p$, we have [3, page 251]

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \operatorname{Im}\frac{\partial}{\partial\theta}(\log f(re^{i\theta}))$$
$$=\operatorname{Re}\frac{(z-p)h'(z)-\overline{(z-p)g'(z)}}{h(z)+\overline{g(z)}} := \operatorname{Re}\frac{A(z)}{B(z)}\le 0.$$

Using the fact that, $\operatorname{Re}[-A(z)/B(z)]\ge 0$ if and only if

$$|1 - A(z)/B(z)| \geq |1 + A(z)/B(z)|, \tag{9}$$

or equivalently, $|A(z) - B(z)| - |A(z) + B(z)| \geq 0.$ We have

$$\begin{aligned}
&|A(z) - B(z)| - |A(z) + B(z)| \\
&= \left|(z-p)h'(z) - \overline{(z-p)g'(z)} - h(z) - \overline{g(z)}\right| \\
&\quad - \left|(z-p)h'(z) - \overline{(z-p)g'(z)} + h(z) + \overline{g(z)}\right| \\
&= \left|-2\alpha + (z-p)\sum_{n=1}^{\infty}(n-1)a_n(z-p)^n - (z-p)\overline{\sum_{n=1}^{\infty}(n+1)b_n(z-p)^n}\right| \\
&\quad - \left|(z-p)\sum_{n=2}^{\infty}(n+1)a_n(z-p)^n - (z-p)^2\overline{\sum_{n=2}^{\infty}(n-1)b_n(z-p)^n}\right| \\
&\geq 2|a| - |z-p|\sum_{n=1}^{\infty}(n-1)|a_n||z-p|^n - |z-p|\sum_{n=1}^{\infty}(n+1)|b_n||z-p|^n \\
&\quad -|z-p|\sum_{n=1}^{\infty}(n+1)|a_n||z-p|^n - |z-p|\sum_{n=1}^{\infty}(n-1)|b_n||z-p|^n \\
&\geq 2\left[|a| - |z-p|^2\sum_{n=1}^{\infty}n(|a_n| + |b_n|)\right] \\
&> 2\left[|\alpha| - (1+p)^2\sum_{n=1}^{\infty}n(|a_n| + |b_n|)\right] \geq 0, \quad \text{by } (8),
\end{aligned}$$

and the result follows. ■

The following theorem contains a distortion estimate.

Theorem 2.9. *If f be of the forms* (3) *and* (5) *and satisfy* (8), *then for* $|z-p| = r < 1-p$

$$\frac{|\alpha|\left[(1+p)^2 - r^2\right]}{(1+p)^2 r} \leq |f(z)| \leq \frac{|\alpha|\left[(1+p)^2 + r^2\right]}{(1+p)^2 r}.$$

Proof. For $0 < |z-p| = r < 1-p$, we see from (3) and (5)

$$\begin{aligned}
|f(z)| &= \left|\frac{\alpha}{z-p} + \sum_{n=1}^{\infty}a_n(z-p)^n + \overline{\sum_{n=1}^{\infty}b_n(z-p)^n}\right| \\
&\geq \frac{1}{|z-p|}\left[|\alpha| - |z-p|\sum_{n=1}^{\infty}(|a_n| + |b_n|)|z-p|^n\right]
\end{aligned}$$

$$\geq \frac{1}{r}\left[|\alpha| - r^2\sum_{n=1}^{\infty}(|a_n|+|b_n|)\right]$$
$$\geq \frac{|\alpha|\left[(1+p)^2 - r^2\right]}{(1+p)^2 r}, \quad \text{by } (8).$$

and

$$\begin{aligned}
|f(z)| &= \left|\frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n(z-p)^n + \overline{\sum_{n=1}^{\infty} b_n(z-p)^n}\right| \\
&\leq \frac{1}{|z-p|}\left[|\alpha| + |z-p|\sum_{n=1}^{\infty}(|a_n|+|b_n|)|z-p|^n\right] \\
&\leq \frac{1}{r}\left[|\alpha| + r^2\sum_{n=1}^{\infty}(|a_n|+|b_n|)\right] \\
&\leq \frac{|\alpha|\left[(1+p)^2 + r^2\right]}{(1+p)^2 r}, \quad \text{by } (8).
\end{aligned}$$

In particular, the disk $\left\{w : |w| < \frac{2|\alpha|p}{(1+p)^2}\right\}$ is contained in $\mathbb{C}\backslash f(U_p)$. ■

Also, denote by $S_{CH}(p)$, the a new subclass of $S'_H(p)$ consisting of functions f of the forms (3) and (4) that are convex.

Theorem 2.10. *Let f be of the forms (3) and (5). If*

$$\sum_{n=1}^{\infty} n^2(|a_n|+|b_n|) \leq \frac{|\alpha|}{(1+p)^2} \tag{10}$$

then f is harmonic univalent, sense preserving in U_p, and $f \in S_{CH}(p)$.

Proof. By using the same method as Theorem 2.8, we obtain that f is univalent and sense preserving in U_p. Also, a function f of the form (3) is said to be convex in U_p if it maps each $0 < |z-p| = r < 1-p$ onto a curve that bounds a convex domain. Such functions f are characterized (see [3]) by $\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f(re^{i\theta})\right\}\right) \leq 0$ if and only if

$$\mathrm{Re}\frac{(z-p)^2h''(z)+(z-p)h'(z) + \overline{(z-p)^2g''(z) + (z-p)g'(z)}}{(z-p)h'(z) - \overline{(z-p)g'(z)}} := \mathrm{Re}\frac{A(z)}{B(z)} \leq 0.$$

Now, using the fact that, $\mathrm{Re}(-A(z)/B(z)) \geq 0$ if and only if $|1 - A(z)/B(z)| \geq |1 + A(z)/B(z)|$, or equivalently, $|A(z) - B(z)| - |A(z) + B(z)| \geq 0$. By using the same method as Theorem 2.8, we obtain that $f \in S_{CH}(p)$. ■

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