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MERIDIAN SURFACES IN $\mathbb{E}^4$ WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. In the present article we study a special class of surfaces in the four-dimensional Euclidean space, which are one-parameter systems of meridians of the standard rotational hypersurface. They are called meridian surfaces. We show that a meridian surface has a harmonic Gauss map if and only if it is part of a plane. Further, we give necessary and sufficient conditions for a meridian surface to have pointwise 1-type Gauss map and find all meridian surfaces with pointwise 1-type Gauss map.

1. Introduction

The study of submanifolds of Euclidean space or pseudo-Euclidean space via the notion of finite type immersions began in the late 1970's with the papers [6, 7] of B.-Y. Chen and has been extensively carried out since then. An isometric immersion $x : M \to \mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ is said to be of *finite type* [6] if x identified with the position vector field of M in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, i.e.,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

where x_0 is a constant map, $x_1, x_2, \dots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are different, then M is said to be of *k-type*. Many results on finite type immersions have been collected in the survey paper [8]. Similarly, a smooth map ϕ of an n-dimensional Riemannian manifold M of $\mathbb{E}^m$ is said to be of finite type if ϕ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of Δ. The notion of finite type immersion is naturally extended to the Gauss map G on M in Euclidean space [10]. Thus, a submanifold M of Euclidean space has *1-type Gauss map* G, if G satisfies $\Delta G = \mu(G+C)$ for some $\mu \in \mathbb{R}$ and some constant vector C (of [2], [3], [4], [13]). However, the Laplacian

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of the Gauss map of some typical well-known surfaces such as the helicoid, the catenoid and the right cone in the Euclidean 3-space $\mathbb{E}^3$ takes a somewhat different form, namely, $\Delta G = \lambda(G + C)$ for some non-constant function λ and some constant vector C. Therefore, it is worth studying the class of surfaces satisfying such an equation. A submanifold M of the Euclidean space $\mathbb{E}^m$ is said to have *pointwise 1-type Gauss map* if its Gauss map G satisfies

$$\Delta G = \lambda(G + C) \tag{1}$$

for some non-zero smooth function λ on M and some constant vector C [11]. A pointwise 1-type Gauss map is called *proper* if the function λ defined by (1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of the *first kind* if the vector C in (1) is zero. Otherwise, the pointwise 1-type Gauss map is said to be of the *second kind* ([9], [11], [14], [15]). In [11] M. Choi and Y. Kim characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with B. Y. Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature [9]. Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map. In [17] D. Yoon studied Vranceanu rotation surfaces in Euclidean 4-space $\mathbb{E}^4$. He obtained classification theorems for the flat Vranceanu rotation surfaces with 1-type Gauss map and an equation in terms of the mean curvature vector [16]. For the general case see [1].

The study of meridian surfaces in the Euclidean 4-space $\mathbb{E}^4$ was first introduced by G. Ganchev and the third author in [12]. The meridian surfaces are one-parameter systems of meridians of the standard rotational hypersurface in $\mathbb{E}^4$. In this paper we investigate the meridian surfaces with pointwise 1-type Gauss map. We give necessary and sufficient conditions for a meridian surface to have pointwise 1-type Gauss map and find all meridian surfaces with pointwise 1-type Gauss map of first and second kind.

2. Preliminaries

In the present section we recall definitions and results of [5]. Let $x : M \to \mathbb{E}^m$ be an immersion from an n-dimensional connected Riemannian manifold M into an m-dimensional Euclidean space $\mathbb{E}^m$. We denote by $\langle , \rangle$ the metric tensor of $\mathbb{E}^m$ as well as the induced metric on M. Let ∇' be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Then the Gauss and Weingarten formulas are given, respectively, by

$$\begin{aligned} \nabla'_X Y &= \nabla_X Y + h(X, Y), \\ \nabla'_X \xi &= -A_\xi X + D_X \xi, \end{aligned}$$

where X, Y are vector fields tangent to M and ξ is a vector field normal to M. Moreover, h is the second fundamental form, D is the linear connection induced in the normal bundle $T^\perp M$, called normal connection, and A_ξ is the shape operator in the direction of ξ that is related with h by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

The covariant differentiation $\overline{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of M is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. The Codazzi equation is given by

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z).$$

We denote by R the curvature tensor associated with ∇, i.e.,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The equations of Gauss and Ricci are given, respectively, by

$$\langle R(X,Y)Z, W\rangle = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle,$$
$$\langle R^\perp(X,Y)\xi, \eta\rangle = \langle [A_\xi, A\eta] X, Y\rangle,$$

for vector fields X, Y, Z, W tangent to M and ξ, η normal to M.

The mean curvature vector field H of an n-dimensional submanifold M in $\mathbb{E}^m$ is given by

$$H = \frac{1}{n}\, trace\, h.$$

A submanifold M is said to be minimal (respectively, totally geodesic) if $H \equiv 0$ (respectively, $h \equiv 0$).

We shall recall the definition of Gauss map G of a submanifold M. Let $G(n,m)$ denotes the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ be the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \binom{m}{n}$. Let $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ be an adapted local orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, \ldots, e_n$, are tangent to M and $e_{n+1}, e_{n+2}, \ldots, e_m$ are normal to M. The map $G : M \to G(n,m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ is called the Gauss map of M. It is a smooth map which carries a point p in M into the oriented n-plane in $\mathbb{E}^m$ obtained by the parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

For any real function ϕ on M the Laplacian of ϕ is defined by

$$\Delta\phi = -\sum_i (\nabla'_{e_i} \nabla'_{e_i} \phi - \nabla'_{\nabla_{e_i} e_i} \phi). \tag{2}$$

3. Classification of meridian surfaces with pointwise 1-type Gauss map

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{E}^4$, and $S^2(1)$ be the 2-dimensional sphere in $\mathbb{E}^3 = span\{e_1, e_2, e_3\}$, centered at the origin O. We consider a smooth curve $c : r = r(v),\ v \in J,\ J \subset \mathbb{R}$ on $S^2(1)$, parameterized by the arc-length ($r'^2(v) = 1$). Let $t(v) = r'(v)$ be the tangent vector field of c. We consider the moving frame field $\{t(v), n(v), r(v)\}$ of the curve c on $S^2(1)$. With respect to this orthonormal frame field the following Frenet formulas hold:

$$\begin{aligned} r' &= t; \\ t' &= \kappa\, n - r; \\ n' &= -\kappa\, t, \end{aligned} \tag{3}$$

where $\kappa(v) = \langle t'(v), n(v)\rangle$ is the spherical curvature of c.

Let $f = f(u),\ g = g(u)$ be non-zero smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $(f'(u))^2 + (g'(u))^2 = 1,\ u \in I$. We consider the surface M^2 in $\mathbb{E}^4$ constructed in the following way:

$$M^2 : z(u, v) = f(u)\, r(v) + g(u)\, e_4, \quad u \in I,\, v \in J \tag{4}$$

(see [12]).

The surface M^2 lies on the rotational hypersurface M^3 in $\mathbb{E}^4$ obtained by the rotation of the meridian curve $\alpha : u \to (f(u), g(u))$ about the Oe_4-axis in $\mathbb{E}^4$. M^2 is called a *meridian surface* on M^3 since it is a one-parameter system of meridians of M^3.

The tangent space of M^2 is spanned by the vector fields:

$$\begin{aligned} z_u &= f' r + g' e_4; \\ z_v &= f\, t, \end{aligned} \tag{5}$$

and hence, the coefficients of the first fundamental form of M^2 are $E = 1;\ F = 0;\ G = f^2(u)$. Taking into account (3) and (5), we calculate the second partial derivatives of $z(u, v)$:

$$\begin{aligned} z_{uu} &= f'' r + g''\, e_4; \\ z_{uv} &= f' t; \\ z_{vv} &= f\kappa\, n - f\, r. \end{aligned}$$

Let us denote $x = z_u,\ y = \frac{z_v}{f} = t$ and consider the following orthonormal normal frame field of M^2:

$$n_1 = n(v); \quad n_2 = -g'(u)\, r(v) + f'(u)\, e_4.$$

Thus we obtain a positive orthonormal frame field $\{x, y, n_1, n_2\}$ of M^2. We denote by κ_α the curvature of the meridian curve α, i.e.,

$$\kappa_\alpha(u) = f'(u)\, g''(u) - g'(u) f''(u).$$

By covariant differentiation with respect to x and y, and a straightforward calculation we obtain

$$(6)\qquad \begin{aligned} \nabla'_x x &= \kappa_\alpha\, n_2;\\ \nabla'_x y &= 0;\\ \nabla'_y x &= \frac{f'}{f}\, y;\\ \nabla'_y y &= -\frac{f'}{f}\, x + \frac{\kappa}{f}\, n_1 + \frac{g'}{f}\, n_2; \end{aligned}$$

and

$$(7)\qquad \begin{aligned} \nabla'_x n_1 &= 0;\\ \nabla'_y n_1 &= -\frac{\kappa}{f}\, y;\\ \nabla'_x n_2 &= -\kappa_\alpha\, x;\\ \nabla'_y n_2 &= -\frac{g'}{f}\, y, \end{aligned}$$

where $\kappa(v)$ and $\kappa_\alpha(u)$ are the curvatures of the spherical c and the meridian curve α, respectively (see [12]).

Equalities (7) imply the following result.

Lemma 3.1. *Let M^2 be a meridian surface given with the surface patch* (4)*. Then*

$$A_{n_1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\kappa}{f} \end{bmatrix}, \qquad A_{n_2} = \begin{bmatrix} \kappa_\alpha & 0 \\ 0 & \frac{g'}{f} \end{bmatrix}.$$

So, the Gauss curvature is given by

$$K = \frac{\kappa_\alpha\, g'}{f}$$

and the mean curvature vector field H of M^2 is

$$H = \frac{\kappa}{2f}\, n_1 + \frac{\kappa_\alpha f + g'}{2f}\, n_2.$$

The Gauss map G of M^2 is defined by $G = x \wedge y$. Using (2), (6), and (7) we calculate that the Laplacian of the Gauss map is expressed as

$$(8)\qquad \begin{aligned} \Delta G = {} & \frac{(f\kappa_\alpha)^2 + \kappa^2 + g'^2}{f^2}\, x \wedge y - \frac{\kappa'}{f^2}\, x \wedge n_1 \\ & -\frac{\kappa f'}{f^2}\, y \wedge n_1 - \frac{f'g' - f(f\kappa_\alpha)'}{f^2}\, y \wedge n_2, \end{aligned}$$

where $\kappa' = \frac{d}{dv}(\kappa)$.

First, we suppose that the Gauss map of M^2 is harmonic, i.e., $\Delta G = 0$. Then from (8) we get

$$\begin{aligned} \kappa_\alpha &= 0; \\ \kappa &= 0; \\ g' &= 0. \end{aligned} \tag{9}$$

So, (6) and (9) imply that M^2 is a totally geodesic surface in $\mathbb{E}^4$. Conversely, if M^2 is totally geodesic, then $\Delta G = 0$.

Thus we obtain the following result.

Theorem 3.2. *Let M^2 be a meridian surfaces in the Euclidean space $\mathbb{E}^4$. The Gauss map of M^2 is harmonic if and only if M^2 is part of a plane.*

Now, we suppose that the meridian surface M^2 is of pointwise 1-type Gauss map, i.e., G satisfies (1), where $\lambda \neq 0$. Then, from equalities (1) and (8) we get

$$\begin{aligned} \lambda + \lambda \langle C, x \wedge y \rangle &= \frac{(f\kappa_\alpha)^2 + \kappa^2 + g'^2}{f^2}; \\ \lambda \langle C, x \wedge n_1 \rangle &= -\frac{\kappa'}{f^2}; \\ \lambda \langle C, y \wedge n_1 \rangle &= -\frac{\kappa f'}{f^2}; \\ \lambda \langle C, y \wedge n_2 \rangle &= -\frac{f'g' - f(f\kappa_\alpha)'}{f^2}. \end{aligned} \tag{10}$$

Using (8) we obtain

$$\begin{aligned} \lambda \langle C, x \wedge n_2 \rangle = 0; \\ \lambda \langle C, n_1 \wedge n_2 \rangle = 0. \end{aligned} \tag{11}$$

Differentiating (11) with respect to u and v we get

$$\begin{aligned} \kappa_\alpha \langle C, x \wedge n_1 \rangle &= 0; \\ \frac{f'}{f} \langle C, y \wedge n_2 \rangle - \frac{g'}{f} \langle C, x \wedge y \rangle &= 0; \\ -\frac{\kappa}{f} \langle C, y \wedge n_2 \rangle + \frac{g'}{f} \langle C, y \wedge n_1 \rangle &= 0. \end{aligned} \tag{12}$$

Since $\lambda \neq 0$ equalities (10) and (12) imply

$$\begin{aligned} &\kappa_\alpha \kappa' = 0; \\ &\kappa \, (f\kappa_\alpha)' = 0; \\ &\lambda f^2 g' = g' \left(1 + (f\kappa_\alpha)^2 + \kappa^2\right) - f f' (f\kappa_\alpha)'. \end{aligned} \tag{13}$$

We distinguish the following cases.

Case I: $g' = 0$. In such case $\kappa_\alpha = 0$. Then equality (8) implies that

$$\Delta G = \frac{\kappa^2}{f^2}\, x \wedge y - \frac{\kappa'}{f^2}\, x \wedge n_1 - \frac{\kappa f'}{f^2}\, y \wedge n_1. \tag{14}$$

If we assume that M^2 has pointwise 1-type Gauss map of the first kind, i.e., $C = 0$, then from (14) we get $\kappa' = 0$ and $\kappa f' = 0$, which imply $\kappa = 0$ since $f' \neq 0$. Hence $\Delta G = 0$, which contradicts the assumption that $\lambda \neq 0$. Consequently, in the case $g' = 0$ there are no meridian surfaces of pointwise 1-type Gauss map of the first kind.

Now we consider meridian surfaces of pointwise 1-type Gauss map of the second kind, i.e., $C \neq 0$. So we suppose that $\kappa \neq 0$. From equalities (1) and (14) we obtain

$$C = \left(\frac{\kappa^2}{\lambda f^2} - 1\right) x \wedge y - \frac{\kappa'}{\lambda f^2}\, x \wedge n_1 - \frac{\kappa f'}{\lambda f^2}\, y \wedge n_1. \tag{15}$$

Using (6), (7) and (15) we obtain

$$\begin{aligned}
\nabla'_x C &= \kappa^2 \left(\frac{1}{\lambda f^2}\right)'_u x \wedge y - \kappa' \left(\frac{1}{\lambda f^2}\right)'_u x \wedge n_1 - \kappa f' \left(\frac{1}{\lambda f^2}\right)'_u y \wedge n_1;\\
\nabla'_y C &= \frac{\kappa}{\lambda^2 f^3}\left(3\kappa'\lambda - \kappa\lambda'_v\right) x \wedge y\\
&\quad + \frac{1}{\lambda^2 f^3}\left(-\kappa''\lambda + k'\lambda'_v + \kappa^3\lambda + \kappa\lambda - \kappa\lambda^2 f^2\right) x \wedge n_1\\
&\quad + \frac{f'}{\lambda^2 f^3}\left(-2\kappa'\lambda + \kappa\lambda'_v\right) y \wedge n_1.
\end{aligned}$$

The last formulas imply that $C = \mathit{const}$ if and only if $\kappa = \mathit{const}$ and $\lambda = \frac{\kappa^2+1}{f^2}$.

The condition $\kappa = \mathit{const} \neq 0$ implies that the curve c on $S^2(1)$ is a circle with non-zero constant spherical curvature. Since $g' = 0$ and $(f'^2 + g'^2) = 1$ we get $f(u) = \pm u + a$, $g(u) = b$, where $a = \mathit{const}$, $b = \mathit{const}$. In this case M^2 is a developable ruled surface. Moreover, from (7) it follows that $\nabla'_x n_2 = 0$; $\nabla'_y n_2 = 0$, which implies that M^2 lies in the 3-dimensional space spanned by $\{x, y, n_1\}$.

Conversely, if $g' = 0$ and $\kappa = \mathit{const}$, by direct computation we get

$$\Delta G = \frac{\kappa^2 + 1}{f^2}(G + C),$$

where $C = -\frac{1}{\kappa^2+1}\, x \wedge y - \frac{\kappa f'}{\kappa^2+1}\, y \wedge n_1$. Hence, M^2 is a surface with pointwise 1-type Gauss map of the second kind.

Summing up we obtain the following result.

Theorem 3.3. *Let M^2 be a meridian surface given with parametrization* (4) *and $g' = 0$. Then M^2 has pointwise 1-type Gauss map of the second kind if and only if the curve c is a circle with non-zero constant spherical curvature* *and the meridian curve α is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = const$, $b = const$. In this case M^2 is a developable ruled surface lying in 3-dimensional space.*

Case II: $g' \neq 0$. In such case from the third equality of (13) we obtain

$$\lambda = \frac{g'\left(1 + (f\kappa_\alpha)^2 + \kappa^2\right) - ff'(f\kappa_\alpha)'}{f^2 g'}. \tag{16}$$

First we shall consider the case of pointwise 1-type Gauss map surfaces of the first kind. From (8) it follows that M^2 is of the first kind ($C = 0$) if and only if

$$\begin{aligned} &\kappa' = 0; \\ &\kappa f' = 0; \\ &f'g' - f(f\kappa_\alpha)' = 0. \end{aligned} \tag{17}$$

The first equality of (17) implies that $\kappa = const$. There are two subcases:

1. $\kappa = 0$. Then the meridian curve α is determined by the equation

$$f'g' - f(f\kappa_\alpha)' = 0. \tag{18}$$

The equalities $\kappa_\alpha = f'g'' - g'f''$ and $f'^2 + g'^2 = 1$ imply that $\kappa_\alpha = -\frac{f''}{g'}$. Hence equation (18) can be rewritten in the form

$$f'\sqrt{1 - f'^2} + f\left(\frac{ff''}{\sqrt{1 - f'^2}}\right)' = 0. \tag{19}$$

Since $\kappa = 0$, M^2 lies in the 3-dimensional space spanned by $\{x, y, n_2\}$.

Conversely, if $\kappa = 0$ and the meridian curve α is determined by a solution $f(u)$ of differential equation (19), the function $g(u)$ is defined by $g' = \sqrt{1 - f'^2}$, then the surface M^2, parameterized by (4), is a surface of pointwise 1-type Gauss map of the first kind.

2. $\kappa \neq 0$. Then the second equality of (17) implies that $f' = 0$. In this case $f(u) = a$; $g(u) = \pm u + b$, where $a = const$, $b = const$. By a result of [12], M^2 is a developable ruled surface in a 3-dimensional space, since $\kappa_\alpha = 0$ and $\kappa = const$. It follows from (16) that $\lambda = \frac{1+\kappa^2}{a^2} = const$, which implies that M^2 has 1-type Gauss map, i.e., M^2 is non-proper. The converse is also true.

Thus we obtain the following result.

Theorem 3.4. *Let M^2 be a meridian surface given with parametrization* (4) *and $g' \neq 0$. Then M^2 has pointwise 1-type Gauss map of the first kind if and only if one of the following holds*:

(i) *the curve c is a great circle on $S^2(1)$ and the meridian curve α is determined by the solutions of the following differential equation*

$$f'\sqrt{1 - f'^2} + f\left(\frac{ff''}{\sqrt{1 - f'^2}}\right)' = 0;$$

(ii) *the curve c is a circle on $S^2(1)$ with non-zero constant spherical curvature and the meridian curve α is determined by $f(u) = a$; $g(u) = \pm u + b$, where $a = const$, $b = const$. In this case M^2 is a developable ruled surface in a 3-dimensional space. Moreover, M^2 is non-proper.*

Now we shall consider the case of pointwise 1-type Gauss map surfaces of the second kind. It follows from equalities (13) that there are three subcases.

1. $\kappa_\alpha = 0$. In this subcase

$$\Delta G = \frac{\kappa^2 + g'^2}{f^2}\, x \wedge y - \frac{\kappa'}{f^2}\, x \wedge n_1 - \frac{\kappa f'}{f^2}\, y \wedge n_1 - \frac{f'g'}{f^2}\, y \wedge n_2. \tag{20}$$

From equalities (1) and (20) we obtain

$$C = \left(\frac{\kappa^2 + g'^2}{\lambda f^2} - 1\right) x \wedge y - \frac{\kappa'}{\lambda f^2}\, x \wedge n_1 - \frac{\kappa f'}{\lambda f^2}\, y \wedge n_1 - \frac{f'g'}{\lambda f^2}\, y \wedge n_2.$$

The third equality in (13) implies that in this case $\lambda = \frac{1+\kappa^2}{f^2}$ and hence, C is expressed as follows:

$$C = -\frac{1}{1+\kappa^2}\left(f'^2\, x \wedge y + \kappa'\, x \wedge n_1 + \kappa f'\, y \wedge n_1 + f'g'\, y \wedge n_2\right). \tag{21}$$

Using (6), (7) and (21) we obtain

$$\begin{aligned}
\nabla'_x C = & -\frac{1}{1+\kappa^2}\left(2f'f''\, x \wedge y + \kappa f''\, y \wedge n_1 + (f'g'' + f''g')\, y \wedge n_2\right);\\
\nabla'_y C = & \frac{1}{f(1+\kappa^2)^2}\Big(\left(2\kappa\kappa' f'^2 + \kappa\kappa'(1+\kappa^2)\right)\, x \wedge y\\
& + \left(2\kappa\kappa'^2 - (1+\kappa^2)\kappa''\right)\, x \wedge n_1\Big)\\
& + \frac{1}{f(1+\kappa^2)^2}\left(-2\kappa' f'\, y \wedge n_1 + 2\kappa\kappa' f'g'\, y \wedge n_2\right).
\end{aligned}$$

The last formulas imply that $C = const$ if and only if $\kappa = const$, $f' = a = const$, $g' = b = const$, $a^2 + b^2 = 1$.

The condition $\kappa = const$ implies that the curve c is a circle on $S^2(1)$. The meridian curve α is given by $f(u) = au + a_1$; $g(u) = bu + b_1$, where $a_1 = const$, $b_1 = const$. In this case M^2 is a developable ruled surface lying in a 3-dimensional space.

Conversely, if $f(u) = au + a_1$; $g(u) = bu + b_1$ and $\kappa = const$, then

$$\Delta G = \frac{\kappa^2 + b^2}{f^2}\, x \wedge y - \frac{\kappa a}{f^2}\, y \wedge n_1 - \frac{ab}{f^2}\, y \wedge n_2.$$

Hence, by direct computation we get

$$\Delta G = \frac{1+\kappa^2}{f^2}(G + C),$$

where $C = -\frac{a}{1+\kappa^2}\left(a\, x \wedge y + \kappa\, y \wedge n_1 + b\, y \wedge n_2\right)$. Consequently, M^2 is a surface of pointwise 1-type Gauss map of the second kind.

2. $\kappa = 0$. In this subcase

$$\Delta G = \frac{(f\kappa_\alpha)^2 + g'^2}{f^2}\, x \wedge y - \frac{f'g' - f(f\kappa_\alpha)'}{f^2}\, y \wedge n_2. \tag{22}$$

From equalities (1) and (22) we obtain

$$C = \left(\frac{(f\kappa_\alpha)^2 + g'^2}{\lambda f^2} - 1\right) x \wedge y - \frac{f'g' - f(f\kappa_\alpha)'}{\lambda f^2}\, y \wedge n_2.$$

Using the third equality of (13) we obtain that C is expressed as follows:

$$C = -\frac{f'g' - f(f\kappa_\alpha)'}{\lambda f^2}\left(\frac{f'}{g'}\, x \wedge y + y \wedge n_2\right), \tag{23}$$

where $\lambda = \frac{1}{f^2}\left(1 + (f\kappa_\alpha)^2 - \frac{ff'}{g'}(f\kappa_\alpha)'\right)$. We denote

$$\varphi = -\frac{f'g' - f(f\kappa_\alpha)'}{\lambda f^2}. \tag{24}$$

Then equalities (6), (7) and (23) imply

$$\begin{aligned} \nabla'_x C &= \left(\left(\varphi\,\frac{f'}{g'}\right)' + \varphi\kappa_\alpha\right) x \wedge y + \left(\varphi' - \varphi\,\frac{f'}{g'}\kappa_\alpha\right) y \wedge n_2; \\ \nabla'_y C &= 0. \end{aligned} \tag{25}$$

It follows from (25) that $C = const$ if and only if $\varphi' = \varphi\,\frac{f'}{g'}\kappa_\alpha$, or equivalently

$$(\ln\varphi)' = \frac{f'}{g'}\kappa_\alpha. \tag{26}$$

Using that $f\kappa_\alpha = -\frac{ff''}{\sqrt{1-f'^2}}$, from (24) we get

$$\varphi = \frac{-\sqrt{1-f'^2}\,\left(f(1-f'^2)(ff'')'^2 f' f''^2 + f'(1-f'^2)^2\right)}{ff'(ff'')'(1-f'^2) + f^2 f''^2 + (1-f'^2)^2}. \tag{27}$$

Now, formulas (26) and (27) imply that $C = const$ if and only if the function $f(u)$ is a solution of the following differential equation

$$\left(\ln\frac{-\sqrt{1-f'^2}\,\left(f(1-f'^2)(ff'')'^2 f' f''^2 + f'(1-f'^2)^2\right)}{ff'(ff'')'(1-f'^2) + f^2 f''^2 + (1-f'^2)^2}\right)' = -\frac{f'f''}{1-f'^2}. \tag{28}$$

Conversely, if $\kappa = 0$ and the meridian curve α is determined by a solution $f(u)$ of differential equation (28), $g(u)$ is defined by $g' = \sqrt{1-f'^2}$, then the surface M^2, parameterized by (4), is a surface of pointwise 1-type Gauss map of the second kind.

3. $\kappa = const \neq 0$ and $f\kappa_\alpha = a = const$, $a \neq 0$. In this subcase

$$\Delta G = \frac{a^2 + \kappa^2 + g'^2}{f^2}\, x \wedge y - \frac{\kappa f'}{f^2}\, y \wedge n_1 - \frac{f'g'}{f^2}\, y \wedge n_2. \tag{29}$$

From equalities (1), (16) and (29) we obtain

$$C = -\frac{1}{1+a^2+\kappa^2}\left(f'^2\, x\wedge y + \kappa f'\, y\wedge n_1 + f'g'\, y\wedge n_2\right). \tag{30}$$

Then equalities (6), (7) and (30) imply

$$\begin{aligned} &\nabla'_x C = -\frac{1}{1+a^2+\kappa^2}\left(f'f''\, x\wedge y + \kappa f''\, y\wedge n_1 + g'f''\, y\wedge n_2\right);\\ &\nabla'_y C = 0. \end{aligned} \tag{31}$$

Formulas (31) imply that $C = const$ if and only if $f'' = 0$. But, if $f'' = 0$, then $\kappa_\alpha = 0$, which contradicts the assumption that $f\kappa_\alpha \neq 0$.

Consequently, if $\kappa = const \neq 0$ and $f\kappa_\alpha = a = const$, $a \neq 0$, then there are no meridian surfaces of pointwise 1-type Gauss map of the second kind.

Summing up we obtain the following result.

Theorem 3.5. *Let M^2 be a meridian surface given with parametrization* (4) *and $g' \neq 0$. Then M^2 has pointwise 1-type Gauss map of the second kind if and only if one of the following holds*:

(i) *the curve c is a circle on $S^2(1)$ and the meridian curve α is determined by $f(u) = au + a_1$; $g(u) = bu + b_1$, where a, a_1, b, b_1 are constants. In this case M^2 is a developable ruled surface lying in a 3-dimensional space*;

(ii) *the curve c is a great circle on $S^2(1)$ and the meridian curve α is determined by the solutions of the following differential equation*

$$\left(\ln\frac{-\sqrt{1-f'^2}\left(f(1-f'^2)(ff'')'^2 f'f''^2 + f'(1-f'^2)^2\right)}{ff'(ff'')'(1-f'^2) + f^2 f''^2 + (1-f'^2)^2}\right)' = -\frac{f'f''}{1-f'^2}.$$

Theorem 3.3, Theorem 3.4, and Theorem 3.5 describe all meridian surfaces with pointwise 1-type Gauss map.

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