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The Riemann Hilbert problem for generalized Q-holomorphic functions

Sezayi Hızlıyel and Mehmet Çağlıyan

Abstract

In this work, the classical Riemann Hilbert boundary value problem is extended to generalized Q-holomorphic functions.

Key Words: Generalized Beltrami systems, Q-Holomorphic functions, Riemann Hilbert problem.

1. Introduction

In [6] A. Douglis developed an analogue of analytic functions theory for more general elliptic systems in the plane of the form

$$w_x + iw_y + aEw_x + bEw_y = 0, \tag{1}$$

where E is an $m \times m$ constant matrix, w is an $m \times 1$ vector, and a and b are complex valued functions of x and y. Subsequently in [5] B. Bojarskiĭ extended the function theory of Douglis to a system which he wrote in the form

$$w_{\overline{z}} = qw_z. \tag{2}$$

He assumed that the variable $m \times m$ matrix q is "lower diagonal with all eigenvalues of q having magnitude less than 1. The systems (1) and (2) are natural ones to consider because they arise from the reduction of general elliptic systems of first order in the plane to a standard canonical form.

Douglis and Bojarskiĭ theory has been used to study the elliptic systems of more general form:

$$w_{\overline{z}} - qw_z = aw + b\overline{w}.$$

Solutions of this equation were called generalized (or pseudo) hyperanalytic functions. Works in this direction appear in [7, 8, 10, 11]. These results extend the generalized (or "pseudo") analytic function theory of Bers [4] and Vekua [17]. Also, the classical boundary value problems for analytic functions were extended to the generalized hyperanalytic functions. A good survey of the methods encountered in the hyperanalytic case may be found in [3, 9], see also [1, 2].

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In [13], Hile noticed that what appears to be the essential property of the elliptic systems in the plane for which one can obtain a useful extension of analytic function theory is the self commuting property of the variable matrix Q, which means

$$Q(z_1)\,Q(z_2) = Q(z_2)\,Q(z_1)$$

for any two points z_1, z_2 in the domain G_0 of Q. Further, such a Q matrix can not be brought into the quasi-diagonal form of Bojarskiĭ by a similarity transformation. So Hile [13] attempts to extend the results of Douglis and Bojarskiĭ to a wider class of systems in the same form as (2). If $Q(z)$ is self-commuting in G_0 and if $Q(z)$ has no eigenvalues of magnitude 1 for each z in G_0, then Hile called the system (2) generalized Beltrami system and the solutions of such a system are called *Q-holomorphic functions.* Later in [14, 15] using Vekua and Bers techniques a function theory is given for the equation

$$w_{\overline{z}} - Qw_z = Aw + B\overline{w}, \tag{3}$$

where the unknown $w(z) = \{w_{ij}(z)\}$ is an $m \times s$ complex matrix, $Q(z) = \{q_{ij}(z)\}$ is a self commuting complex matrix with $m \times m$, and $q_{k,k-1} \neq 0$ for $k = 2, \ldots m$. $A = \{a_{ij}(z)\}$ and $B = \{b_{ij}(z)\}$ are $m \times m$-complex matrices commuting with Q. Solutions of such equation were called *generalized Q-holomorphic functions.*

In this work, we consider the Riemann Hilbert boundary value problem for the equation (3) with the boundary condition

$$Re(\overline{\gamma}w) = \varphi \quad on\ \partial G,$$

where the coefficients A and B are Hölder continuous in a bounded simply connected region G with piecewise Hölder continuous boundary. γ is commuting with Q. A and B are continuous in $G \cup \partial G$. Moreover, γ has one Hölder continuous derivative, φ is real Hölder continuous function on ∂G. Also we assume that Q commute with $\overline{Q}$.

2. Fundamental operators

To investigate Q-holomorphic functions, Hile introduced the notion of generating solution for generalized Beltrami operator

$$D := \frac{\partial}{\partial \overline{z}} - Q\frac{\partial}{\partial z}. \tag{4}$$

This generating solution can be written as $\phi(z) := \phi_0(z)\,I + N(z)$ and satisfies the equation $D\phi = 0$, where N is the nilpotent part of ϕ and ϕ_0 is the main diagonal term of ϕ satisfying the Beltrami equation

$$\frac{\partial \phi_0}{\partial \overline{z}} - \lambda\frac{\partial \phi_0}{\partial z} = 0,$$

where $|\lambda(z)| \neq 1$.

Hile also gave the following representation formula called the generalized Cauchy-Pompieu representation for the $m \times s$ complex matrix-valued functions.

Theorem 1 *Let G be a regular subdomain of G_0, with $\Gamma = \partial G$ and w be an $m \times s$-matrix in $C^1(G) \cap C(\overline{G})$ with bounded first derivatives in G. Then for z in G*

$$\begin{aligned} w(z) &= P^{-1}\int_{\Gamma} (\phi(\zeta) - \phi(z))^{-1} d\phi(\zeta)\, w(\zeta) \\ &\quad -2iP^{-1}\iint_{G} \phi_\zeta(\zeta)(\phi(\zeta) - \phi(z))^{-1}\left[w_{\overline{\zeta}}(\zeta) - Q(\zeta)\, w_\zeta(\zeta)\right] d\xi d\eta. \end{aligned} \tag{5}$$

In (5) P is constant matrix defined by

$$P(z) = \int_{|z|=1} (zI + \overline{z}Q)^{-1}(Idz + Qd\overline{z}). \tag{6}$$

It is called P-value for (4) [13].

Using Beltrami homeomorphism $\rho(z) = \phi_0(z)$, we may write

$$\frac{\partial}{\partial \overline{z}} - Q\frac{\partial}{\partial z} = \left[\overline{\rho_z}(\overline{\lambda}Q - I)\right]\left(\frac{\partial}{\partial \overline{\rho}} - \widehat{Q}\frac{\partial}{\partial \rho}\right),$$

where $\widehat{Q} = \left[\overline{\rho_z}(\overline{\lambda}Q - I)\right]^{-1}\left[\rho_z(\lambda I - Q)\right]$ is self-commuting matrix whose the main diagonal terms are zero (see [14], pp. 431). Note that for the equation in normal form the generating solution is $\phi(z) = zI + N(z)$ and the complex Pompieu formula is

$$w(z) = P^{-1}\int_{\partial G}\frac{w(\zeta)}{\zeta - z}d\zeta - 2iP^{-1}\iint_{G}\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z}d\xi d\eta, \tag{7}$$

where $N(z)$ is $m \times m$-type nilpotent matrix (see [16], pp. 581). The operators

$$\begin{aligned} \widetilde{\Phi}_{\partial G} w(z) &= P^{-1}\int_{\partial G}\frac{w(\zeta)}{\zeta - z}d\zeta \\ \widetilde{T}_G w(z) &= -2iP^{-1}\iint_{G}\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z}d\xi d\eta \end{aligned}$$

and

$$\widetilde{\Pi}_G w(z) = -2iP^{-1}\iint_{G}\frac{w_{\overline{\zeta}}(\zeta)}{(\zeta - z)^2}d\xi d\eta$$

have similar properties as Φ, T and Π-operators of Vekua's theory and the following theorems concerning $\widetilde{\Phi}_{\partial G}$, $\widetilde{T}_G$ and $\widetilde{\Pi}_G$ can be proved as in the book of Vekua [17].

Theorem 2 *If $G \in C_\alpha^m$, then $\widetilde{\Phi}_{\partial G} : C_\alpha^m(\partial G) \to C_\alpha^m(\overline{G})$.*

Theorem 3 *If $f \in L_1(G)$, then $\partial_{\overline{z}}\widetilde{T}_G f = f$. If $p > 2$, then $\widetilde{T}_G : L_p(\overline{G}) \to C_{\frac{p-2}{p}}(\mathbb{C})$. If $G \in C, w \in C(\overline{G})$, and $\partial_{\overline{z}} w \in L_p(\overline{G}), p > 2$, then*

$$w(z) = \widetilde{\Phi}_{\partial G} w(z) + \widetilde{T}_G(\partial_{\overline{\zeta}} w)(z).$$

Theorem 4 *If $G \in C_\alpha^{m+1}$, then $\widetilde{T}_G : C_\alpha^m(\overline{G}) \to C_\alpha^{m+1}(\overline{G})$ and $\widetilde{\Pi}_G : C_\alpha^m(\overline{G}) \to C_\alpha^m(\overline{G})$. Moreover $\partial_z \widetilde{T}_G = \widetilde{\Pi}_G$. The operator $\widetilde{\Pi}_G$ can be extended to a bounded linear operator on $L_p(\mathbb{C}), p > 1$, with $\partial_z \widetilde{T}_{\mathbb{C}} = \widetilde{\Pi}_{\mathbb{C}}$.*

As in the complex case (see [12], pp. 259), using the complex Pompieu formula, for any complex matrix-valued function w that is $C^1(G)$ and Hölder continuous in $\overline{G}$ we have the representation

$$w(z) = \Omega(z) + (\mathbf{P}f)(z), \quad w_{\overline{z}}(z) = f(z),$$

where

$$\begin{aligned}
\Omega(z) &= -2\pi i P^{-1} \int_{\partial G} \left[d_n G^I(\zeta, z) - i d G^{II}(\zeta, z) \right] Rew(\zeta) \\
&\quad -2\pi P^{-1} \int_{\partial G} d_n G^{II}(\zeta, z) \operatorname{Im} w(\zeta) \\
(\mathbf{P}f)(z) &= -2\pi P^{-1} \iint_G \left[G_\zeta^I(\zeta, z) + G_\zeta^{II}(\zeta, z) \right] f(\zeta)\, d\zeta d\overline{\zeta} \\
&\quad -2\pi P^{-1} \iint_G \left[G_{\overline{\zeta}}^I(\zeta, z) + G_{\overline{\zeta}}^{II}(\zeta, z) \right] \overline{f(\zeta)} d\zeta d\overline{\zeta}.
\end{aligned} \tag{8}$$

G^I and G^{II} are the first and second Green's functions for G and d_n denotes the differential in normal direction. If θ is a conformal mapping of $\overline{G}$ onto the unit disk $\mathbb{C}_0$, then the Green functions of first and second kinds may be expressed as

$$\begin{aligned}
G^I(\zeta, z) &:= \frac{-1}{2\pi} \log \left| \frac{\theta(\zeta) - \theta(z)}{1 - \overline{\theta(\zeta)}\theta(z)} \right| \\
G^{II}(\zeta, z) &:= \frac{-1}{2\pi} \log \left| (\theta(\zeta) - \theta(z)) \left(1 - \overline{\theta(\zeta)}\theta(z) \right) \right|.
\end{aligned}$$

Thus $\mathbf{P}f$ has the representation

$$(\mathbf{P}f)(z) = P^{-1} \iint_G \frac{\theta'(\zeta) f(\zeta)}{\theta(\zeta) - \theta(z)} d\zeta d\overline{\zeta} + P^{-1}\theta(z) \iint_G \frac{\overline{\theta'(\zeta) f(\zeta)}}{1 - \overline{\theta(\zeta)}\theta(z)} d\zeta d\overline{\zeta} \tag{9}$$

and $\frac{\partial}{\partial z}(\mathbf{P}f)(z)$ may be expressed as

$$\begin{aligned}\frac{\partial\left(\mathbf{P}f\right)(z)}{\partial z} &= \left(\widetilde{\Pi}f\right)(z)\\ &= P^{-1}\theta'(z)\iint_G\left\{\frac{\theta'(\zeta)f(\zeta)}{\left[\theta(\zeta)-\theta(z)\right]^2}+\frac{\overline{\theta'(\zeta)f(\zeta)}}{\left[1-\overline{\theta(\zeta)}\theta(z)\right]^2}\right\}d\zeta d\overline{\zeta}.\end{aligned}$$

The operator $\mathbf{P}$ may still put in a more convenient form. If we introduce the inverse mapping $z=\rho(t):=\theta^{-1}(t)$, we have

$$(\mathbf{P}f)(z)=\left(\widetilde{T}_{\mathbb{C}_0}f(\rho)\rho'\right)(\theta(z))-\theta(z)\left(\widetilde{T}_{\mathbb{C}\setminus\mathbb{C}_0}\overline{f_1(\rho_1)\rho_1'}\right)(\theta(z))$$

where $\mathbb{C}_0$ is the unit disk and

$$\begin{aligned}\rho_1(z) &:= \overline{\rho\left(\frac{1}{\overline{z}}\right)}, \quad (z\in\mathbb{C}\setminus\mathbb{C}_0)\\ f_1(\rho_1(z)) &:= \frac{1}{\overline{z}}f\left(\overline{\rho_1(z)}\right).\end{aligned}$$

By writing $\mathbf{P}f$ in above form we obtain certain imbedding properties of $\widetilde{T}$ and $\widetilde{\Pi}$

$$C_\alpha\left(\mathbf{P}f,\overline{G}\right)\le M_1(\alpha,G)\,C_\alpha\left(f,\overline{G}\right) \tag{10}$$

$$C_\alpha\left(\Pi f,\overline{G}\right)\le M_1(\alpha,G)\,C_\alpha\left(f,\overline{G}\right). \tag{11}$$

3. The Riemann-Hilbert problem

We consider the problem

$$Dw=Aw+B\overline{w}\ \ in\ \ G,\quad Re\left(\overline{\gamma}w\right)=\varphi\ \ on\ \ \partial G. \tag{12}$$

where the coefficients A and B are Hölder continuous in a bounded simply connected region G with piecewise Hölder continuous boundary. $\gamma=\gamma_0 I+N(z)$ is commuting with Q. A and B are continuous in $G\cup\partial G$. Moreover, γ has one Hölder continuous derivative and φ is real Hölder continuous function on ∂G. We assume Q commute with $\overline{Q}$. It is natural for us define the index of this problem as

$$\kappa:=\mathrm{ind}\overline{\gamma}:=\frac{1}{2\pi}\int_{\partial G}d\arg\overline{\gamma}_0.$$

Case 1. $\kappa=0$. In the case of index zero, we may reduce our problem by setting $\omega=\overline{\gamma}w$ to the case

$$\begin{aligned}D\omega &= \widetilde{A}\omega+\widetilde{B}\overline{\omega}\quad in\quad G,\\ Re\omega &= \varphi\quad on\quad \partial G,\end{aligned}$$

where

$$\widetilde{A} = A + \overline{\gamma}^{-1} D\overline{\gamma}, \quad \widetilde{B} = \overline{\gamma} B \gamma^{-1}.$$

To see that such a transformation is valid it is necessary to demonstrate that the inverse γ^{-1} exits in G. This follows directly from the fact that $|\gamma_0| \neq 0$ on ∂G, and hence it is possible to continue harmonically the component of $\gamma := R \ \exp(-i\theta)$ into interior of G such that $R_0 := |\gamma_0|$ nowhere vanishes. Our problem may be written as the system

$$\begin{aligned} D\omega_{k\ell} &= \widetilde{A}_{kk}\omega_{k\ell} + \widetilde{B}_{kk}\overline{\omega}_{k\ell} + f_{k\ell} \quad in \ \ G \\ \mathrm{Re}\omega_{k\ell} &= \varphi_{k\ell} \quad on \quad \partial G, \quad k = 1, \cdots, m, \quad \ell = 1, \cdots, s \end{aligned} \tag{13}$$

where

$$f_{1\ell} = 0, \quad f_{k\ell} = \sum_{j=1}^{k-1} \left(q_{kj} \frac{\partial \omega_{j\ell}}{\partial z} + \widetilde{A}_{kj}\omega_{j\ell} + \widetilde{B}_{kj}\overline{\omega}_{j\ell} \right), \quad (2 \le k \le m, \ 1 \le \ell \le s).$$

The problem (13), which may be solved successively, may be replaced by integral equation by using Green functions $G^I(\zeta, z)$, $G^{II}(\zeta, z)$ of first and second kinds respectively. We obtain

$$\omega_{k\ell} = \Omega_{k\ell} + \mathbf{P}\left(\widetilde{A}_{kk}\omega_{k\ell} + \widetilde{B}_{kk}\overline{\omega}_{k\ell} + f_{k\ell} \right),$$

where $\Omega_{k\ell}$ is an analytic function given by

$$\Omega_{k\ell} = -2\pi P^{-1} \int_{\partial G} \left[d_n G^I(\zeta, z) - i d G^{II}(\zeta, z) \right] \varphi_{k\ell} + c_{k\ell} \tag{14}$$

and $c_{k\ell}$ is arbitrary constant which can be fixed by setting it equal to boundary norm,

$$c_{k\ell} = 2\pi P^{-1} \int_{\partial G} \mathrm{Im}(\omega_{k\ell}) \, d_n G^{II}(\zeta, z), \quad k = 1, \cdots, m, \quad \ell = 1, \cdots, s.$$

Taking advantage of the fact that Q is nilpotent yields a concise representation for ω as

$$\omega = \Omega + \mathbf{P}\left[\sum_{k=0}^{m-1} \left(Q\widetilde{\Pi} \right)^k \left(\widetilde{A}\omega + \widetilde{B}\overline{\omega} + Q\Omega' \right) \right],$$

where

$$\Omega = \sum_{k=1}^{m} \sum_{\ell=1}^{s} \Omega_{k\ell} e^{k\ell}, \quad \Omega' = \frac{\partial \Omega}{\partial z} \quad \text{and} \quad \widetilde{\Pi} = \frac{\partial \mathbf{P}}{\partial z}.$$

and $e^{k\ell}$ denotes $m \times s$ constant matrix in which k − th row and ℓ − th column terms are 1 and the others terms are 0. Moreover, by introducing

$$\mathbf{R} := \mathbf{P}\left(I - Q\widetilde{\Pi} \right)^{-1} = \mathbf{P} \sum_{k=0}^{m-1} \left(Q\widetilde{\Pi} \right)^k, \tag{15}$$

we obtain the matrix integral equation

$$\omega = \Omega - \mathbf{R}\left(Q\Omega^{'}\right) + \mathbf{R}\left(\widetilde{A}\omega + \widetilde{B}\overline{\omega}\right). \tag{16}$$

Using imbedding properties (10) and (11) we have following imbedding property in $C_\alpha(\overline{G})$

$$C_\alpha\left(\mathbf{R}f, \overline{G}\right) \leq M_1(\alpha, G)\frac{M_3^m(\alpha, G) - 1}{M_3(\alpha, G) - 1} C_\alpha\left(f, \overline{G}\right), \tag{17}$$

where

$$M_3(\alpha, G) := (m-1) M_2(\alpha, G) C_\alpha\left(Q, \overline{G}\right), \tag{18}$$

and the norm $C_\alpha(f, \overline{G})$ is

$$C_\alpha(f, \overline{G}) = \sum_{l=1}^{m}\sum_{\ell=1}^{s} C_\alpha\left(f_{k\ell}, \overline{G}\right).$$

The operator $\mathbf{R}$ is then seen from (15), (17) and (18) to be compact in C_α, hence is a Fredholm integral operator. To show that integral equation (16) has a unique solution we consider the homogenous version of (12) i.e. $\mathrm{Re}\omega\mid_{\partial G} = 0$, $c = 0$. This means, since $Re\Omega \mid_{\partial G} = 0$, that $\Omega = 0$ and the Fredholm integral equation corresponding to homogenous Riemann Hilbert problem is homogenous integral equation

$$\omega = \mathbf{R}\left(\widetilde{A}\omega + \widetilde{B}\overline{\omega}\right).$$

It is easily seen that this homogenous integral equation has only trivial solution. This discussion is then summarized as

Theorem 5 *For any given real, $m \times s$ matrix-valued function $\varphi \in C_\alpha(\partial G)$ and given real, $m \times s$-type constant matrix c there exits a unique solution w of the integral equation (16) which satisfies*

$$\int\limits_{\partial G} Im(\omega)\, d_n G^{II}(\zeta, z) = c.$$

Case 2. $\kappa < 0$. We assume in present case the index is a negative integer

$$\kappa := \mathrm{ind}\overline{\gamma} := \frac{1}{2\pi}\int_{\partial G} d\arg\overline{\gamma}_0 = -n.$$

We introduce a transformation

$$v := \psi^{-1}w, \quad \psi = \prod_{\tau=1}^{n}\left[\phi(z) - \phi(z_\tau)\right],$$

where the points z_τ lie in G. The boundary value problem becomes

$$Dv = \widehat{A}v + \widehat{B}\overline{v} \;\; in \;\; G, \quad \mathrm{Re}\left(\overline{\rho}v\right) = \varphi \;\; \text{on} \;\; \partial G,$$

where

$$\widehat{A} = A, \quad \widehat{B} = \psi^{-1} B \overline{\psi} \quad and \quad \rho = \gamma \overline{\psi}.$$

This reduced Riemann Hilbert problem has index zero, and we have modified the problem to case previously discussed.

The homogenous boundary value problem for v which is normalized such that

$$c_h = -2\pi i P^{-1} \int_{\partial G} \mathrm{Im} v_h(\zeta) d_n G^{II}(\zeta, z)$$

has non-trivial solution v_h. That exits a non-trivial solution to this problem can be seen by considering $1-$th row and $\ell-$th column of v_h

$$\begin{aligned} \frac{\partial v_{h1\ell}}{\partial \overline{z}} &= \widehat{A}_{11} v_{h1\ell} + \widehat{B}_{11} \prod_{\tau=1}^{n} \frac{\overline{z - z_\tau}}{z - z_\tau} \overline{v_{h1\ell}} \\ \mathrm{Re}\left(\overline{\rho_{11}} v_{h1\ell}\right) &= 0\,, \quad \mathrm{Ind}\overline{\rho_{11}} = 0 \end{aligned}$$

which is known to have a solution non vanishing in $\overline{G}$ (see [12], 11.1).

Now, for fixed ℓ, $1 \le \ell \le s$, we consider the boundary value problem

$$\begin{aligned} Dw^\ell &= Aw^\ell + B\overline{w^\ell} \quad in \quad G \\ \mathrm{Re}\left(\overline{\gamma} w^\ell\right) &= \varphi^\ell \quad on \quad \partial G, \end{aligned} \tag{19}$$

where $w^\ell = \sum_{i=1}^{m} w_{i\ell} e^{i\ell}$ and $\varphi^\ell = \sum_{i=1}^{m} \varphi_{i\ell} e^{i\ell}$ are $m \times s$ matrix-valued functions. Hence the homogenous solutions w_h^ℓ of (19) with index $\kappa = -n$ has a representation of the form

$$w_h^\ell = \lambda_\ell \psi v_h^\ell,$$

where λ_ℓ is a real constant matrix commuting with Q and $v_h^\ell = \sum_{i=1}^{m} v_{hi\ell} e^{i\ell}$ is a solution of

$$Dv^\ell = \widehat{A} v^\ell + \widehat{B} \overline{v^\ell}, \quad \mathrm{Re}(\overline{\rho} v^\ell) = 0.$$

Note that if λ commutes with Q then λ can be written as

$$\lambda = \sum_{k=1}^{m} P_k \lambda_{k1},$$

where $P_1 = I, P_k = \sum_{1=1}^{m} \sum_{j=1}^{i-1} \left(C_{ij}\right)_k e^{ij}, (2 \le k \le m)$ and $\left(C_{k,k-l}\right)_\mu$ are real or complex constants such that

$$\begin{aligned} \left(C_{k1}\right)_\mu &= \begin{cases} 1, & \mu = k \\ 0, & \mu \ne k \end{cases} \\ \left(C_{k,k-1}\right)_2 &= \frac{q_{k,k-1}}{q_{21}} \end{aligned}$$

$$(C_{k,k-l})_\mu$$

$$: \ = \begin{cases} \dfrac{1}{(C_{k-l,k-l-1})_2} \sum\limits_{s=1}^{l-1} \begin{vmatrix} (C_{k,k-s})_2 & (C_{k,k-s})_\mu \\ (C_{k-s,k-l-1})_2 & (C_{k-s,k-l-1})_\mu \end{vmatrix} & \text{for } \mu = 3, \ldots, l+1 \\ 0 & \text{for } \mu > l+1 \end{cases}$$

$$\begin{aligned} (C_{k,k-l})_2 &= \frac{q_{k,k-l}}{q_{21}} - \frac{1}{(C_{k-l,k-l-1})_2} \sum_{\mu=3}^{l+1} (C_{k,k-l})_\mu \frac{q_{\mu 1}}{q_{21}} \\ &= \frac{a_{k,k-l}}{a_{21}} - \frac{1}{(C_{k-l,k-l-1})_2} \sum_{\mu=3}^{l+1} (C_{k,k-l})_\mu \frac{a_{\mu 1}}{a_{21}} \end{aligned}$$

(see [14], pp.442). Moreover, it is clear that λ_1 and λ_2 commute with Q then λ_1 commutes with λ_2.

From this it is easily seen that each homogenous solutions of (19) having $n+1$ distinct zeros must be vanish identically in G. The general solution to (12) may be written as $w = \psi(v_0 + \sum_{\ell=1}^{s} \lambda_\ell v_h^\ell)$, where v_0 is a particular solution of reduced equation and λ_ℓ are real constant matrices commuting with Q.

If w_1^ℓ, w_2^ℓ are distinct solutions of

$$\begin{aligned} Dw^\ell &= Aw^\ell + B\overline{w^\ell} \quad \text{in} \quad G, \\ \text{Re}\overline{\gamma} w^\ell &= 0 \quad on \quad \partial G \end{aligned} \tag{20}$$

with negative index, then any combination of them

$$w = \lambda_1 w_1^\ell + \lambda_2 w_2^\ell$$

with real constant matrices λ_1 and λ_2 commuting with Q is also a solution of (20). The general solution of (20) contains $2n$ arbitrary real constant $z_\tau = x_\tau + iy_\tau$, $(1 \le \tau \le n)$ and an arbitrary real constant matrix λ_ℓ. It may therefore be conjectured that there are $2n+1$ linearly independent solution of (20).

r solutions $w_1^\ell, \cdots, w_r^\ell$ of (20) are said to be linearly independent if the equation

$$\sum_{j=1}^{r} \lambda_j w_j^\ell = 0, \quad (\lambda_j \ \text{ commuting } \ \text{ with } \ Q)$$

implies that $\lambda_j = 0$.

Suppose that we already know $(2n+1)$ linearly independent solutions $\widetilde{w}_0^\ell, \widetilde{w}_1^\ell, \cdots, \widetilde{w}_{2n}^\ell$ of (20). No pair of these solutions can have the same zeros. To show that there are no more than $(2n+1)$ solutions, with non vanishing $1-$th row and $\ell-$th column terms , we show that each such solution can be written as a linear combination of $\widetilde{w}_j^\ell$, $(0 \le j \le 2n)$. To this end let

$$\lambda_\mu^{(0)} := \sum_{i=1}^{m} \sum_{j=1}^{i} \lambda_{\mu ij}^{(0)} e^{ij}$$

be non-trivial solution of the system

$$\sum_{\mu=0}^{2n} \lambda_{\mu}^{(0)} \widetilde{w}_{\mu}^{\ell}(z_\tau) = 0, \quad 1 \le \tau \le n.$$

In component form this becomes

$$\sum_{\mu=0}^{2n} \sum_{j=1}^{i} \lambda_{\mu i j}^{(0)} \widetilde{w}_{\mu j \ell}(z_\tau) = 0, \quad (1 \le i \le m,\ 1 \le \ell \le s,\ 1 \le \tau \le n)$$

For each μ we define the functions

$$w_{\mu}^{\ell} := \sum_{k=1}^{2n} \lambda_k^{(\mu)} \widetilde{w}_k^{\ell}(z),$$

where $\lambda_k^{(\mu)}$ are real constant matrices commuting with Q that are uniquely determined as the solution of system

$$w_{2\mu}^{\ell}(z_\tau) = \delta_{\mu\tau} e^{1\ell} \quad , \quad w_{2\mu-1}^{\ell}(z_\tau) = i\delta_{\mu\tau} e^{1\ell}.$$

In complex form this becomes

$$\begin{aligned} \sum_{k=1}^{2n} \sum_{j=1}^{t} \lambda_{ktj}^{(2\mu)} w_{kj\ell}(z_\tau) &= \delta_{1t}\delta_{\mu\tau}, \\ \sum_{k=1}^{2n} \sum_{j=1}^{t} \lambda_{ktj}^{(2\mu-1)} w_{kj\ell}(z_\tau) &= i\delta_{1t}\delta_{\mu\tau}, \end{aligned}$$

$1 \le t \le m$, $1 \le \mu, \tau \le n$. To show that these inhomogeneous equations have a unique solution it is sufficient to demonstrate that the system

$$\sum_{k=1}^{2n} \chi_k w_k^{\ell}(z_\tau) = 0$$

possesses only trivial solution, where χ_k commutes with Q. This follows directly the fact that if

$$\sum_{k=1}^{2n} \chi_k \widetilde{w}_k^{\ell}(z_\tau)$$

is a solution of (20), that is, linearly independent of w_0^{ℓ}, it must be trivial solution. From this we conclude, since $\widetilde{w}_k^{\ell}$ are linearly independent, that the χ_k are all zero. In this way we construct a system of functions w_{μ}^{ℓ}, $\mu = 0, \cdots, 2n$ satisfying the conditions

$$w_0^{\ell}(z_\tau) = 0, \quad w_{2\mu}^{\ell}(z_\tau) = \delta_{\mu\tau} e^{1\ell}, \quad w_{2\mu-1}^{\ell}(z_\tau) = i\delta_{\mu\tau} e^{1\ell},$$

$1 \le t \le m$, $1 \le \mu, \tau \le n$. It is seen that each solution of (20) may be represented as a linear combination of $\widetilde{w}_{\mu}^{\ell}$. For instance if we set $(\lambda_{2\tau})_{k1} = Re w_{k\ell}(z_\tau)$ and $(\lambda_{2\tau-1})_{k1} = \mathrm{Im} w_{k\ell}(z_\tau)$ then the function $w^{\ell} - \sum_{\mu=1}^{2n} \lambda_\mu w_{\mu}^{\ell}$

is a solution of (20) and it vanishes at z_τ, $\tau = 1, \cdots, n$. Where $(\cdots)_{k1}$ means the $k-$ th row and $1-$ th column elements of $(\cdots)$. Therefore there is a unique real constant matrix commuting with Q such that

$$w^\ell(z) - \sum_{\mu=1}^{2n} \lambda_\mu w_\mu^\ell(z) = \lambda_0 w_0^\ell(z).$$

We can, moreover, show that there are at most $(2n+1)$ linearly independent solutions. To this end, we introduce $w_{2\mu,\tau}^\ell$ and $w_{2\mu-1,\tau}^\ell$ as linear combinations of $\widetilde{w}^\ell$:

$$\begin{aligned} w_{2\mu,t}^\ell &:= \sum_{k=1}^{2n} \lambda_k^{(2\mu,t)} \widetilde{w}_k^\ell \\ w_{2\mu-1,t}^\ell &:= \sum_{k=1}^{2n} \lambda_k^{(2\mu-1,t)} \widetilde{w}_k^\ell \end{aligned}$$

which moreover satisfy the conditions

$$w_{2\mu,t}^\ell(z_\tau) := \delta_{\mu\tau} e^{t\ell}, \quad w_{2\mu-1,t}^\ell(z_\tau) := i\delta_{\mu\tau} e^{t\ell}.$$

These two conditions for determining $\lambda_k^{(2\mu,t)}$, $\lambda_k^{(2\mu-1,t)}$ may be formulated in terms of their components as

$$\begin{aligned} \sum_{k=1}^{2n}\sum_{j=1}^{l} \lambda_{klj}^{(2\mu,t)} \widetilde{w}_{kj\ell}(z_\tau) &= \delta_{tl}\delta_{\mu\tau} \\ \sum_{k=1}^{2n}\sum_{j=1}^{l} \lambda_{klj}^{(2\mu-1,t)} \widetilde{w}_{kj\ell}(z_\tau) &= i\delta_{tl}\delta_{\mu\tau} \end{aligned}$$

$1 \le t, l \le m$, $1 \le \mu, \tau \le n$. If we fix $w_{0,t}^\ell := (w_0)_{t\ell}$ $(1 \le t \le m)$, then each solution of (20) may be written as a linear combination of $w_{\mu,t}^\ell(z)$.

Now we show that there are exactly $(2n+1)$ linearly independent solutions of (20). Let $w_0^\ell(z)$ be non-trivial solution of (20) that vanishes at each of given points z_τ $(1 \le \tau \le n)$. If we have $2n$ additional solutions that moreover satisfy the conditions

$$w_{2k}^\ell(z_\tau) = \delta_{k\tau} e^{1\ell}, \quad w_{2k-1}^\ell(z_\tau) = i\delta_{k\tau} e^{1\ell}, \quad 1 \le k, \tau \le n,$$

then these solutions also have non-vanishing $1-$th row and $\ell-th$ column terms and form a linearly independent system with $w_0(z)$.

Let us define

$$f_k := \prod_{\substack{\tau=1 \\ \tau \ne k}}^{n} \left[\phi(z) - \phi(z_\tau)\right].$$

Then if w^ℓ is a solution of (20), $v^\ell = f_k^{-1} w^\ell$ is a solution of the homogenous boundary value problem of index -1,

$$Dv^\ell = Av^\ell + f_k^{-1} B\overline{f_k}\overline{v^\ell} \ \ in \ \ G, \ \operatorname{Re}\left(\overline{\gamma} f_k v^\ell\right) = 0 \ \ on \ \ \partial G. \tag{21}$$

If v_a^ℓ, v_b^ℓ are two non-trivial solutions of (21) that satisfy the boundary conditions $\mathrm{Re}v_a^\ell = \mathrm{Im}v_b^\ell = 0$ on ∂G, then it is known (see [12], pp.272) that without loss of generality we may assume that $\mathrm{Im}v_{a1\ell} > 0$, $\mathrm{Re}v_{b1\ell} > 0$ on ∂G. Furthermore let $v_\alpha^\ell(z)$, $v_\beta^\ell(z)$ be two solutions of (21) that satisfy $v_\alpha^\ell(z_\tau) = v_\beta^\ell(z_\tau) = 0$ and the inhomogeneous boundary conditions

$$\begin{aligned} \mathrm{Re}\left(\overline{\gamma} f_k v_\alpha^\ell\right) &= -\mathrm{Re}\left(\overline{\gamma} f_k v_a^\ell\right) \quad on \quad \partial G \\ \mathrm{Re}\left(\overline{\gamma} f_k v_\beta^\ell\right) &= -\mathrm{Re}\left(\overline{\gamma} f_k v_b^\ell\right) \quad on \quad \partial G. \end{aligned}$$

Then the two functions $\omega_1^\ell := f_k(v_a^\ell + v_\alpha^\ell)$, $\omega_2^\ell := f_k(v_b^\ell + v_\beta^\ell)$ are solutions of (20) that satisfy

$$\begin{aligned} \omega_1^\ell(z_\tau) &= \omega_2^\ell(z_\tau) = 0, \quad \tau \neq k, \quad 1 \leq \tau \leq n \\ \omega_1^\ell(z_k) &= f_k(z_k)\, v_a^\ell(z_k) \neq 0, \quad \omega_2^\ell(z_k) = f_k(z_k)\, v_b^\ell(z_k) \neq 0. \end{aligned}$$

Consequently, one has $\mathrm{Im}\left[(\omega_1)_{1\ell}(z_k)\,\overline{(\omega_2)_{1\ell}(z_k)}\right] \neq 0$ and the linear equations

$$\begin{aligned} \lambda_{2n}^{(1)}\omega_1^\ell(z_k) + \lambda_{2n}^{(2)}\omega_2^\ell(z_k) &= e^{1\ell} \\ \lambda_{2n-1}^{(1)}\omega_1^\ell(z_k) + \lambda_{2n-1}^{(2)}\omega_2^\ell(z_k) &= ie^{1\ell} \end{aligned}$$

may be solved for real constant matrices $\lambda_{2n}^{(1)}$, $\lambda_{2n}^{(2)}$, $\lambda_{2n-1}^{(1)}$, $\lambda_{2n-1}^{(2)}$, commuting with Q, having non vanishing main diagonal terms. In this way we may construct two solutions

$$w_\mu^\ell = \lambda_\mu^{(1)}\omega_1^\ell + \lambda_\mu^{(2)}\omega_2^\ell\,, \quad \mu = 2n-1,\ 2n$$

with the properties

$$w_{2k}^\ell(z_\tau) = \delta_{k\tau}e^{1\ell}\,, \quad w_{2k-1}^\ell(z_\tau) = i\delta_{k\tau}e^{1\ell}.$$

By doing this for each k, we obtain $2n+1$ linearly independent solutions. Hence, the homogenous boundary value problem (20) has exactly $(2n+1)m$ linearly independent solutions over $\mathbb{R}$. This discussion is then summarized as the next theorem.

Theorem 6 *The homogenous boundary value problem*

$$Dw = Aw + B\overline{w} \quad in \quad G, \qquad Re\left(\overline{\gamma} w\right) = 0 \quad on \quad \partial G$$

has exactly $(2n+1)ms$ linearly independent solutions with non-identically vanishing $1-th$ row and $\ell - th$ column terms of w over $\mathbb{R}$.

Case 3. $\kappa > 0$. We consider the boundary value problem

$$Dw = Aw + B\overline{w} \quad in \quad G, \quad \mathrm{Re}\left(\overline{\gamma} w\right) = \varphi \quad on \quad \partial G.$$

We assume in present case the index is a positive integer

$$\kappa := \mathrm{Ind}\overline{\gamma} = \frac{1}{2\pi}\int_{\partial G} d\arg\overline{\gamma}_0 = n.$$

We introduce a transformation

$$\omega = \psi w, \quad \psi^{-1} := \prod_{\tau=1}^{\kappa} \left[\phi(z) - \phi(z_\tau)\right]^{-1},$$

where the points z_τ lie in G. Then the new boundary value problem becomes

$$D\omega = \widetilde{A}\omega + \widetilde{B}\overline{\omega} \quad in \quad G, \quad \mathrm{Re}\left(\overline{\gamma}\psi^{-1}\omega\right) = \varphi \quad on \quad \partial G,$$

where

$$\widetilde{A} = A, \quad \widetilde{B} = \psi B \overline{\psi^{-1}}.$$

Each of non-trivial solutions of homogenous boundary value problem with integral index has no zeros on the boundary ∂G. Since $\overline{w_{1\ell}}$ and $\overline{\gamma}_0$ are perpendicular on ∂G, $\overline{w_{1\ell}}$ has the same index as $\overline{\gamma_0}$. So, with each solution w^ℓ of differential equation

$$Dw^\ell = Aw^\ell + B\overline{w^\ell}$$

which has no zeros on ∂G, i.e. $w \neq 0$ on ∂G, we can always associate a homogenous boundary value problem of integral index. Hence the index

$$n = \frac{1}{2\pi} \int_{\partial G} d\arg \overline{w_{1\ell}}$$

of a solution w^ℓ of homogenous differential equation

$$Dw^\ell = Aw^\ell + B\overline{w^\ell},$$

which has neither zeros nor poles on ∂G, is equal to the difference between the numbers of its poles and zeros in G. Hence every such solution w has a representation of the form

$$w = \sum_{\ell=1}^{s} \lambda_\ell \psi w_h^\ell$$

and the poles and zeros of ψ coincide with those of w.

We clearly do not investigate the conditions under which there are continuous solution in $\overline{G}$ even when the boundary vector family has positive index

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Sezayi HIZLIYEL, Mehmet ÇAĞLIYAN
Uludağ University, Art and Science Faculty
Deparment of Mathematics
16059 Görükle-Bursa, TURKEY
e-mail : hizliyel@uludag.edu.tr

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