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Neighborhoods of a new class of harmonic multivalent functions

Elif Yaşar, Sibel Yalçın*

Department of Mathematics, Faculty of Arts and Science, Uludag University, Bursa, Turkey

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ABSTRACT

We introduce and investigate a new subclass of harmonic multivalent functions defined by using a differential operator. We obtain coefficient conditions, distortion bounds, extreme points, convex combination for the above class of harmonic multivalent functions. We also, derive inclusion relationships involving the neighborhoods of harmonic multivalent functions.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain D we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. The harmonic function $f = h + \overline{g}$ is sense preserving and locally one to one in D if $|h'(z)| > |g'(z)|$ in D. See Clunie and Sheil-Small [1].

For $p \geq 1$, $n \in \mathbb{N}$, denote by $SH\ (n, p)$ the class of functions $f = h + \overline{g}$ that are sense preserving, harmonic multivalent in the unit disk $U = \{z : |z| < 1\}$, where h and g defined by

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k,\ |b_{n+p-1}| < 1, \tag{1.1}$$

which are analytic and multivalent functions in U.

Note that $SH(n, p)$ reduces to $S(n, p)$, the class of analytic multivalent functions, if the co-analytic part of $f = h + \overline{g}$ is identically zero.

Let $f^{(q)}$ denote the qth-order ordinary differential operator, for a function $f \in SH\ (n, p)$, that is,

$$f^{(q)}(z) = h^{(q)}(z) + \overline{g^{(q)}(z)}$$

where $h^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} a_k\ z^{k-q}$ and $g^{(q)}(z) = \sum_{k=n+p-1}^{\infty} \frac{k!}{(k-q)!} b_k\ z^{k-q}$, $p > q, p \in \mathbb{N}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, n \in \mathbb{N}, z \in U$.

Next, $D^m f^{(q)}(z)$ is defined by

$$D^m f^{(q)}(z) = D^m h^{(q)}(z) + (-1)^m \overline{D^m g^{(q)}(z)} \tag{1.2}$$

* Corresponding author.
E-mail addresses: elifyasar@uludag.edu.tr (E. Yaşar), syalcin@uludag.edu.tr, skarpuz@uludag.edu.tr (S. Yalçın).
URL: http://www20.uludag.edu.tr/~syalcin/ (S. Yalçın).

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where

$$D^m h^{(q)}(z) = \frac{(p-q)^m}{(p-q)!} p! z^{p-q} + \sum_{k=n+p}^{\infty} \frac{(k-q)^m}{(k-q)!} k! a_k z^{k-q} \quad \text{and}$$

$$D^m g^{(q)}(z) = \sum_{k=n+p-1}^{\infty} \frac{(k-q)^m}{(k-q)!} k! b_k z^{k-q}, \quad m \in \mathbb{N}_0, z \in U.$$

In view of (1.2), it is clear that

$$D^0 f^{(0)}(z) = h(z) + \overline{g(z)}$$
$$D^0 f^{(1)}(z) = h^{'}(z) + \overline{g^{'}(z)}$$
$$D^1 f^{(0)}(z) = zh^{'}(z) - \overline{zg^{'}(z)}.$$

For $q = 0$, the differential operator $D^m f^{(q)}$ was introduced for the class $S(1,1)$ by Salagean [2] and modified for the class $SH(1,1)$ by Jahangiri et al. [3].

We will use the notations

$$\frac{(i-q)^j}{(i-q)!} i! := q_i^j, \quad i = p, k, n+p, n+p-1 \text{ and } j = m, m+1.$$

Let

$$F(z) = (1-\lambda) D^m f^{(q)}(z) + \lambda D^{m+1} f^{(q)}(z) = H(z) + \overline{G(z)} (f(z) \in SH(n,p), 0 \leq \lambda \leq 1)$$

where H and G are of the form

$$H(z) = (1-\lambda+\lambda(p-q)) q_p^m z^{p-q} + \sum_{k=n+p}^{\infty} (1-\lambda+\lambda(k-q)) q_k^m a_k z^{k-q},$$

$$G(z) = (-1)^m \sum_{k=n+p-1}^{\infty} ((1-\lambda) - \lambda(k-q)) q_k^m b_k z^{k-q}. \tag{1.3}$$

Let $SH_{n,p}^m(q,\lambda,\alpha)$ denote the subclass of $SH(n,p)$ consisting of functions $f = h + \overline{g} \in SH(n,p)$ that satisfy the condition

$$\operatorname{Re}\left\{ \frac{zH^{'}(z) - \overline{zG^{'}(z)}}{H(z) + \overline{G(z)}} \right\} > \alpha(p-q), \quad (0 \leq \alpha < 1, p > q, p \in \mathbb{N}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, n \in \mathbb{N}, z \in U) \tag{1.4}$$

where $H(z)$ and $G(z)$ are given by (1.3).

Denote by $\overline{SH}(n,p)$ the subclass of $SH(n,p)$, consisting of harmonic functions $f_m = h + \overline{g_m}$ where h and g_m are of the form

$$h(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad g_m(z) = (-1)^m \sum_{k=n+p-1}^{\infty} b_k z^k, \quad a_k, b_k \geq 0. \tag{1.5}$$

Define $\overline{SH}_{n,p}^m(q,\lambda,\alpha) := SH_{n,p}^m(q,\lambda,\alpha) \cap \overline{SH}(n,p)$.

The classes $SH_{n,p}^m(q,\lambda,\alpha)$ and $\overline{SH}_{n,p}^m(q,\lambda,\alpha)$ include a variety of well-known subclasses of $SH(n,p)$. For example,

(i) $SH_{1,1}^0(0,0,0) \equiv SH^*$ is the class of sense-preserving, harmonic univalent functions f which are starlike in U (see [4,5]);

(ii) $\overline{SH}_{1,1}^0(0,0,\alpha) \equiv SH^*(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U (see [6]);

(iii) $\overline{SH}_{1,1}^1(0,0,\alpha) \equiv HK(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in U (see [4]);

(iv) $\overline{SH}_{1,p}^0(0,0,0) \equiv SH^*(p)$ is the class of sense-preserving, harmonic multivalent functions which are starlike in U (see [7]);

(v) $\overline{SH}_{1,1}^m(0,0,\alpha) \equiv \overline{H}(m,\alpha)$ is the class of sense preserving, Salagean-type harmonic univalent functions in U (see [3]);

(vi) $\overline{SH}_{1,1}^m(0,0,\alpha) \equiv \overline{S}_H(m+1,m;\alpha)$ (see [8]).

If the co-analytic part of $f = h + \overline{g}$ of the form (1.1) is identically zero and specialize the parameters, we obtain the following subclasses:

(i) $\overline{SH}_{n,p}^m(q,0,\alpha) \equiv S_{n,p}^m(q,(1-\alpha)(p-q),1)$ (see [9]);

(ii) $\overline{SH}_{n,p}^0(q,0,\alpha) \equiv S_{n,p}^q(0,1,(1-\alpha)(p-q))$ (see [10]).

2. Properties of the class $\overline{SH}^m_{n,p}(q,\lambda,\alpha)$

Denote by $SH^*(n,p)$ the class of functions $f_m = h + \overline{g}_m$ for $m = 0$ of the form (1.5) which are sense preserving and multivalent harmonic starlike, satisfying the condition $\frac{\partial}{\partial\theta}(\arg f_m(re^{i\theta})) \geq 0$, for each $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $0 \leq r < 1$.

Lemma 1. *Let $f_m = h + \overline{g}_m$ for $m = 0$ be of the form (1.5). Then $f_m \in SH^*(n,p)$ if and only if*

$$\sum_{k=n+p}^{\infty} ka_k + \sum_{k=n+p-1}^{\infty} kb_k \leq p \quad (p \geq 1, n \in \mathbb{N}). \tag{2.1}$$

Theorem 1. *Let $f = h + \overline{g}$ be given by (1.1). Furthermore, let*

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{[((1-\alpha)(p-q)+1)-|(1-\alpha)(p-q)-1|](1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m}|a_k|$$
$$+ \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))|(1-\lambda)-\lambda(k-q)|}{[((1-\alpha)(p-q)+1)-|(1-\alpha)(p-q)-1|](1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m}|b_k| \leq \frac{1}{2},$$
$$\times (0 \leq \alpha < 1, p > q, p \in \mathbb{N}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, n \in \mathbb{N}, z \in U) \tag{2.2}$$

then f is sense preserving, harmonic multivalent in U, and $f \in SH^m_{n,p}(q,\lambda,\alpha)$.

Proof. If the inequality (2.2) holds for the coefficients of $f = h + \overline{g}$, then by (2.1), f is sense preserving and harmonic multivalent in U. In view of (1.4), we need to prove that $\mathrm{Re}\{w\} > 0$, where

$$w = \frac{zH'(z) - \overline{zG'(z)} - \alpha(p-q)[H(z) + \overline{G(z)}]}{H(z) + \overline{G(z)}} := \frac{A(z)}{B(z)}.$$

Using the fact that $\mathrm{Re}w > 0 \Leftrightarrow |1+w| > |1-w|$, it suffices to show that

$$|A(z)+B(z)| - |A(z)-B(z)| > 0.$$

Therefore we obtain

$$|A(z)+B(z)| - |A(z)-B(z)|$$
$$\geq [((1-\alpha)(p-q)+1)-|(1-\alpha)(p-q)-1|](1-\lambda+\lambda(p-q))q_p^m|z|^{p-q}$$
$$- \sum_{k=n+p}^{\infty} 2(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))q_k^m|a_k||z|^{k-q}$$
$$- \sum_{k=n+p-1}^{\infty} 2(k-q+\alpha(p-q))|(1-\lambda)-\lambda(k-q)|q_k^m|b_k||z|^{k-q}$$
$$> [((1-\alpha)(p-q)+1)-|(1-\alpha)(p-q)-1|](1-\lambda+\lambda(p-q))q_p^m|z|^{p-q}$$
$$\times \left\{1 - \sum_{k=n+p}^{\infty} \frac{2(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{[((1-\alpha)(p-q)+1)-|(1-\alpha)(p-q)-1|](1-\lambda+\lambda(p-q))}\frac{q_k^m}{q_p^m}|a_k|\right.$$
$$\left. - \sum_{k=n+p-1}^{\infty} \frac{2(k-q+\alpha(p-q))|(1-\lambda)-\lambda(k-q)|}{[((1-\alpha)(p-q)+1)-|(1-\alpha)(p-q)-1|](1-\lambda+\lambda(p-q))}\frac{q_k^m}{q_p^m}|b_k|\right\}.$$

This last expression is non-negative by (2.2), and so the proof is complete. □

Corollary 1. *For $\lambda < \frac{1}{n+p-q}$ and $\alpha \geq 1 - \frac{1}{p-q}$, if the inequality*

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))}\frac{q_k^m}{q_p^m}|a_k|$$
$$+ \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))}\frac{q_k^m}{q_p^m}|b_k| \leq (1-\alpha)(p-q)$$
$$\times (0 \leq \alpha < 1, p > q, p \in \mathbb{N}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

holds, then $f \in SH^m_{n,p}(q,\lambda,\alpha)$.

Corollary 2. *For $\lambda < \frac{1}{n+p-q}$ and $\alpha \le 1 - \frac{1}{p-q}$, if the inequality*

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} |a_k| + \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} |b_k| \le 1 \times (0 \le \alpha < 1, p > q, p \in \mathbb{N}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

holds, then $f \in SH_{n,p}^m(q, \lambda, \alpha)$.

Theorem 2. *Let $f_m = h + \overline{g_m}$ be given by (1.5). Also, suppose that $\lambda < \frac{1}{n+p-q}$. Then*

(i) *for $\alpha \ge 1 - \frac{1}{p-q}$, $f_m \in \overline{SH}_{n,p}^m(q, \lambda, \alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k + \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k \le (1-\alpha)(p-q); \tag{2.3}$$

(ii) *for $\alpha \le 1 - \frac{1}{p-q}$, $f_m \in \overline{SH}_{n,p}^m(q, \lambda, \alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k + \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k \le 1. \tag{2.4}$$

Proof. The "if" part follows from Theorem 1, Corollaries 1 and 2 upon noting that $\overline{SH}_{n,p}^m(q, \lambda, \alpha) \subset SH_{n,p}^m(q, \lambda, \alpha)$. For the "only if" part, we show that $f_m \in \overline{SH}_{n,p}^m(q, \lambda, \alpha)$ if the condition (2.4) does not hold.

Note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (1.5), to be in $\overline{SH}_{n,p}^m(q, \lambda, \alpha)$ is that the condition (1.4) to be satisfied. This is equivalent to

$$\mathrm{Re}\left\{ \frac{(1-\alpha)(p-q)z^{p-q} - \sum_{k=n+p}^{\infty} (k-q-\alpha(p-q)) \frac{(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k z^{k-q}}{z^{p-q} - \sum_{k=n+p}^{\infty} \frac{(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k z^{k-q} + (-1)^{2m} \sum_{k=n+p-1}^{\infty} \frac{((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k \overline{z^{k-q}}} \times \frac{-(-1)^{2m} \sum_{k=n+p-1}^{\infty} (k-q+\alpha(p-q)) \frac{((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^{m+1}}{q_p^m} b_k \overline{z^{k-q}}}{z^{p-q} - \sum_{k=n+p}^{\infty} \frac{(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k z^{k-q} + (-1)^{2m} \sum_{k=n+p-1}^{\infty} \frac{((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k \overline{z^{k-q}}} \right\} \ge 0. \tag{2.5}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$ we must have

$$\frac{(1-\alpha)(p-q) - \sum_{k=n+p}^{\infty} (k-q-\alpha(p-q)) \frac{(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k r^{k-p}}{1 - \sum_{k=n+p}^{\infty} \frac{(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k r^{k-p} + \sum_{k=n+p-1}^{\infty} \frac{((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k r^{k-p}} \times \frac{- \sum_{k=n+p-1}^{\infty} (k-q+\alpha(p-q)) \frac{((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^{m+1}}{q_p^m} b_k r^{k-p}}{1 - \sum_{k=n+p}^{\infty} \frac{(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k r^{k-p} + \sum_{k=n+p-1}^{\infty} \frac{((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k r^{k-p}} \ge 0. \tag{2.6}$$

If the condition (2.4) does not hold then the numerator of (2.6) is negative for r sufficiently close to 1 because of conditions (i) or (ii). Thus there exists a $z_0 = r_0$ in (0, 1) for which the quotient in (2.6) is negative. This contradicts the required condition for $f \in \overline{SH}^m_{n,p}(q, \lambda, \alpha)$ and so the proof is complete. □

Next we determine the distortion bounds for the functions in $\overline{SH}^m_{n,p}(q, \lambda, \alpha)$.

Theorem 3. *Let $f_m \in \overline{SH}^m_{n,p}(q, \lambda, \alpha)$. Also, suppose that $\lambda < \frac{1}{n+p-q}$. Then for $|z| = r < 1$ we have*

(i) *for $\alpha \geq 1 - \frac{1}{p-q}$,*

$$|f_m(z)| \leq (1 + b_{n+p-1})r^p + \left(\frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} - \frac{(n-1+(1+\alpha)(p-q))(1-\lambda(n+p-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} b_{n+p-1}\right) r^{n+p},$$

and

$$|f_m(z)| \geq (1 - b_{n+p-1})r^p - \left(\frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} - \frac{(n-1+(1+\alpha)(p-q))(1-\lambda(n+p-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} b_{n+p-1}\right) r^{n+p},$$

(ii) *for $\alpha \leq 1 - \frac{1}{p-q}$,*

$$|f_m(z)| \leq (1 + b_{n+p-1})r^p + \left(\frac{(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} - \frac{(n-1+(1+\alpha)(p-q))(1-\lambda(n+p-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} b_{n+p-1}\right) r^{n+p},$$

and

$$|f_m(z)| \geq (1 - b_{n+p-1})r^p - \left(\frac{(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} - \frac{(n-1+(1+\alpha)(p-q))(1-\lambda(n+p-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} b_{n+p-1}\right) r^{n+p}.$$

Proof. (i) We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_m \in \overline{SH}^m_{n,p}(q, \lambda, \alpha)$. Taking the absolute value of f_m we have

$$\begin{aligned}
|f_m(z)| &\leq (1 + b_{n+p-1})r^p + \sum_{k=n+p}^{\infty} (a_k + b_k) r^k \\
&\leq (1 + b_{n+p-1})r^p + \sum_{k=n+p}^{\infty} (a_k + b_k) r^{n+p} \\
&= (1 + b_{n+p-1})r^p + \frac{(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} \\
&\quad \times \sum_{k=n+p}^{\infty} \frac{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m}{(1-\lambda+\lambda(p-q))q_p^m} (a_k + b_k) r^{n+p} \\
&\leq (1 + b_{n+p-1})r^p + \frac{(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} \\
&\quad \times \sum_{k=n+p}^{\infty} \left(\frac{(k-q-\alpha(p-q))((1-\lambda)+\lambda(k-q))q_k^m}{(1-\lambda+\lambda(p-q))q_p^m} a_k\right.
\end{aligned}$$

$$+ \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))q_k^m}{(1-\lambda+\lambda(p-q))q_p^m} b_k \Bigg) r^{n+p}.$$

Using Theorem 2(i), we obtain

$$|f_m(z)| \le (1+b_{n+p-1})r^p + \left(\frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} - \frac{(n-1+(1+\alpha)(p-q))(1-\lambda(n+p-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} b_{n+p-1} \right) r^{n+p}.$$

The proof of other case is similar and so is omitted. □

The following covering result follows from the left hand inequality in Theorem 3.

Corollary 3. *Let f_m of the form (1.5) be such that $f_m \in \overline{SH}_{n,p}^m(q,\lambda,\alpha)$ and $\lambda < \frac{1}{n+p-q}$. Then*

(i) *for* $\alpha \ge 1 - \frac{1}{p-q}$,

$$\left\{ w : |w| < \left[1 - \frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} - \frac{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m + (n-1+(1+\alpha)(p-q))(1-\lambda(n+p-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} b_{n+p-1} \right] \right\} \subset f(U),$$

(ii) *for* $\alpha \le 1 - \frac{1}{p-q}$,

$$\left\{ w : |w| < \left[1 - \frac{(1-\lambda+\lambda(p-q))q_p^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} - \frac{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m + (n-1+(1+\alpha)(p-q))(1-\lambda(n+p-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))((1-\lambda)-\lambda(n+p-q))q_{n+p}^m} b_{n+p-1} \right] \right\} \subset f(U).$$

Theorem 4. *Let f_m be given by (1.5) and $\lambda < \frac{1}{n+p-q}$. Then $f_m \in clco\overline{SH}_{n,p}^m(q,\lambda,\alpha)$ if and only if*

$$f_m(z) = \sum_{k=n+p-1}^{\infty} (x_k h_k(z) + y_k g_{m_k}(z)),$$

where $h_{n+p-1}(z) = z^p$, $h_k(z)$, for $k = n+p, n+p+1, \ldots$ is of the form

$$h_k(z) = \begin{cases} z^p - \dfrac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))} \dfrac{q_p^m}{q_k^m} z^k; & \alpha \ge 1 - \dfrac{1}{p-q} \\ z^p - \dfrac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))} \dfrac{q_p^m}{q_k^m} z^k; & \alpha \le 1 - \dfrac{1}{p-q} \end{cases}$$

and $g_{m_k}(z)$, for $k = n+p-1, n+p, \ldots$ is of the form

$$g_{m_k}(z) = \begin{cases} z^p + (-1)^m \dfrac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))} \dfrac{q_p^m}{q_k^m} \bar{z}^k; & \alpha \ge 1 - \dfrac{1}{p-q} \\ z^p + (-1)^m \dfrac{(1-\lambda+\lambda(p-q))}{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))} \dfrac{q_p^m}{q_k^m} \bar{z}^k; & \alpha \le 1 - \dfrac{1}{p-q} \end{cases}$$

$$x_{n+p-1} \equiv x_p = 1 - \left(\sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k \right), \quad x_k \ge 0,\ y_k \ge 0.$$

In particular, the extreme points of $\overline{SH}_{n,p}^m(q,\lambda,\alpha)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof. Suppose $\alpha \geq 1 - \frac{1}{p-q}$ and

$$\begin{aligned} f_m(z) &= \sum_{k=n+p-1}^{\infty} (x_k h_k(z) + y_k g_{m_k}(z)) \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))} \frac{q_p^m}{q_k^m} x_k z^k \\ &\quad + (-1)^m \sum_{k=n+p-1}^{\infty} \frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))} \frac{q_p^m}{q_k^m} y_k \bar{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k \left(\frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))} \frac{q_p^m}{q_k^m} x_k \right) \\ &\quad + \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k \left(\frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))} \frac{q_p^m}{q_k^m} y_k \right) \\ &= \sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k = 1 - x_p \leq 1, \end{aligned}$$

and so $f_m \in \overline{SH}_{n,p}^m(q, \lambda, \alpha)$.

Conversely, if $f_m \in \overline{SH}_{n,p}^m(q, \lambda, \alpha)$, then

$$a_k \leq \frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))} \frac{q_p^m}{q_k^m}$$

and

$$b_k \leq \frac{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))}{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))} \frac{q_p^m}{q_k^m}.$$

Set

$$\begin{aligned} x_k &= \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_k, \quad (k = n+p, n+p+1, \ldots) \\ y_k &= \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(p-q)(1-\alpha)(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_k. \quad (k = n+p-1, n+p, \ldots) \end{aligned}$$

and

$$x_p = 1 - \left(\sum_{k=n+p}^{\infty} x_k + \sum_{k=n+p-1}^{\infty} y_k \right)$$

where $x_p \geq 0$. Then, as required, we obtain

$$f_m(z) = x_p z^p + \sum_{k=n+p}^{\infty} x_k h_k(z) + \sum_{k=n+p-1}^{\infty} y_k g_{m_k}(z).$$

The proof for the case $\alpha \leq 1 - \frac{1}{p-q}$ is similar and hence is omitted. □

Theorem 5. *The class $\overline{SH}_{n,p}^m(q, \lambda, \alpha)$ is closed under convex combinations.*

Proof. Let $\lambda < \frac{1}{n+p-q}$ and $f_{m_i} \in \overline{SH}_{n,p}^m(q, \lambda, \alpha)$ for $i = 1, 2, 3, \ldots$, where f_{m_i} is given by

$$f_{m_i}(z) = z^p - \sum_{k=n+p}^{\infty} a_{k_i} z^k + (-1)^m \sum_{k=n+p-1}^{\infty} b_{k_i} \overline{z^k}.$$

Then by (2.3) and (2.4),

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha\,(p-q))\,(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_{k_i} + \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_{k_i}$$
$$\leq \begin{cases} (1-\alpha)(p-q) & \text{if } \alpha \geq 1-\dfrac{1}{p-q} \\ 1 & \text{if } \alpha \leq 1-\dfrac{1}{p-q}. \end{cases} \tag{2.7}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z^p - \sum_{k=n+p}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + (-1)^m \sum_{k=n+p-1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i}\right) \bar{z}^k.$$

Then by (2.7),

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} \left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) \\
&\quad + (-1)^m \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} \left(\sum_{i=1}^{\infty} t_i b_{k_i}\right) \\
&= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=n+p}^{\infty} \frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} a_{k_i} \right. \\
&\quad \left. + (-1)^m \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha(p-q))((1-\lambda)-\lambda(k-q))}{(1-\lambda+\lambda(p-q))} \frac{q_k^m}{q_p^m} b_{k_i} \right\} \\
&\leq \begin{cases} (1-\alpha)(p-q) \displaystyle\sum_{i=1}^{\infty} t_i = (1-\alpha)(p-q) & \text{if } \alpha \geq 1-\dfrac{1}{p-q} \\ 1 \displaystyle\sum_{i=1}^{\infty} t_i = 1 \quad \text{if } \alpha \leq 1-\dfrac{1}{p-q}. \end{cases}
\end{aligned}$$

This is the condition required by (2.3) and (2.4) and so $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \overline{SH}_{n,p}^m(q,\lambda,\alpha)$. □

Theorem 6. *Let* $\lambda < \frac{1}{n+p-q}$ *and* $\alpha_1 \geq 1 - \frac{1}{p-q}$. *For* $\alpha_1 < \alpha_2$,

$$\overline{SH}_{n,p}^m(q,\lambda,\alpha_2) \subset \overline{SH}_{n,p}^m(q,\lambda,\alpha_1).$$

Proof. Let $\lambda < \frac{1}{n+p-q}$, $\alpha_1 \geq 1 - \frac{1}{p-q}$, $\alpha_1 < \alpha_2$ and $f_m(z) \in \overline{SH}_{n,p}^m(q,\lambda,\alpha_2)$.

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha_1(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))(1-\alpha_1)} \frac{q_k^m}{q_p^{m+1}} |a_k| \\
&\quad + \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha_1(p-q))|(1-\lambda)-\lambda(k-q)|}{(1-\lambda+\lambda(p-q))(1-\alpha_1)} \frac{q_k^m}{q_p^{m+1}} |b_k| \\
&\leq \sum_{k=n+p}^{\infty} \frac{(k-q-\alpha_2(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))(1-\alpha_2)} \frac{q_k^m}{q_p^{m+1}} |a_k| \\
&\quad + \sum_{k=n+p-1}^{\infty} \frac{(k-q+\alpha_2(p-q))|(1-\lambda)-\lambda(k-q)|}{(1-\lambda+\lambda(p-q))(1-\alpha_2)} \frac{q_k^m}{q_p^{m+1}} |b_k| \\
&\leq 1,
\end{aligned}$$

then $f_m(z) \in \overline{SH}_{n,p}^m(q,\lambda,\alpha_1)$. □

3. Neighborhoods for the class $\overline{SH}^{m}_{n,p}(q,\lambda,\alpha)$

Following Goodman [11] and Ruscheweyh [12] (see also [13,14,10], [9]), we define the set of the δ-neighborhood of $f=h+\overline{g}\in\overline{SH}(n,p)$,

$$N^{\delta}_{n,p}(f^{(q)}_m;g^{(q)}_m)=\Bigg\{g_m\in\overline{SH}(n,p):g_m(z)=z^p-\sum_{k=n+p}^{\infty}A_kz^k+(-1)^m\sum_{k=n+p-1}^{\infty}B_k\overline{z}^k,A_k,B_k\geq 0,B_{n+p-1}<1,\text{ and }$$

$$\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}k(|a_k-A_k|+|b_k-B_k|)+\frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}|b_{n+p-1}-B_{n+p-1}|\leq\delta,\delta>0\Bigg\}. \quad (3.1)$$

In particular, for the function $h(z)=z^p$, we immediately have

$$N^{\delta}_{n,p}(h^{(q)};g^{(q)}_m)=\Bigg\{g_m\in\overline{SH}(n,p):g_m(z)=z^p-\sum_{k=n+p}^{\infty}A_kz^k+(-1)^m\sum_{k=n+p-1}^{\infty}B_k\overline{z}^k,$$

$$A_k,B_k\geq 0,B_{n+p-1}<1,\text{ and }\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}k\,(A_k+B_k)+\frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}|B_{n+p-1}|\leq\delta,\delta>0\Bigg\}. \quad (3.2)$$

Theorem 7. *Let $\lambda<\frac{1}{n+p-q}$. If $g_m(z)\in\overline{SH}^{m}_{n,p}(q,\lambda,\alpha)$, then*

$$\overline{SH}^{m}_{n,p}(q,\lambda,\alpha)\subset N^{\delta}_{n,p}(h^{(q)};g^{(q)}_m),$$

where $h(z)$ and $g_m(z)$ are given by (3.2),

(i) *for $\alpha\geq 1-\frac{1}{p-q}$,*

$$\delta=\frac{(n+p)(1-\alpha)(p-q)}{(n+(1-\alpha)(p-q))\times\Psi}-\left(\frac{(n+p)(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q^m_{n+p-1}}{(n+(1-\alpha)(p-q))(1-\lambda-\lambda(n+p-q))(n+p-q)^m}-\frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}\right)B_{n+p-1},$$

(ii) *for $\alpha\leq 1-\frac{1}{p-q}$,*

$$\delta=\frac{(n+p)}{(n+(1-\alpha)(p-q))\times\Psi}-\left(\frac{(n+p)(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q^m_{n+p-1}}{(n+(1-\alpha)(p-q))(1-\lambda-\lambda(n+p-q))(n+p-q)^m}-\frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}\right)B_{n+p-1},$$

and

$$\Psi=\left(\frac{(1-\lambda-\lambda(n+p-q))(n+p-q)^m}{((1-\lambda)+\lambda(p-q))q^m_p}\right).$$

Proof. Let $g_m(z)\in\overline{SH}^{m}_{n,p}(q,\lambda,\alpha)$ and $\alpha\geq 1-\frac{1}{p-q}$. We need to show that $g_m(z)\in N^{\delta}_{n,p}(h^{(q)};g^{(q)}_m)$. It suffices to show that g_m satisfies the condition (3.2). In view of Theorem 2 (i), we have

$$\Psi\times\left[\sum_{k=n+p}^{\infty}(k-q-\alpha(p-q))\frac{k!}{(k-q)!}A_k+\sum_{k=n+p}^{\infty}(k-q+\alpha(p-q))\frac{k!}{(k-q)!}B_k\right]$$
$$\leq(1-\alpha)(p-q)-\frac{(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q^m_{n+p-1}}{((1-\lambda)+\lambda(p-q))q^m_p}B_{n+p-1}.$$

Then,

$$\begin{aligned}\sum_{k=n+p}^{\infty}\left(\frac{k!}{(k-q)!}k\right)(A_k+B_k) &\le \frac{(1-\alpha)(p-q)}{\Psi}\\ &\quad-\frac{(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q_{n+p-1}^{m}}{\Psi\times((1-\lambda+\lambda(p-q))q_p^m)}B_{n+p-1}\\ &\quad+(q+\alpha(p-q))\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}(A_k+B_k)\\ &\le \frac{(1-\alpha)(p-q)}{\Psi}-\frac{(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q_{n+p-1}^{m}}{(1-\lambda-\lambda(n+p-q))(n+p-q)^m}B_{n+p-1}\\ &\quad+\frac{(q+\alpha(p-q))}{n+p}\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}k(A_k+B_k),\end{aligned}$$

so that,

$$\begin{aligned}\sum_{k=n+p}^{\infty}\left(\frac{k!}{(k-q)!}k\right)(A_k+B_k) \le\ &\frac{(n+p)(1-\alpha)(p-q)}{(n+(1-\alpha)(p-q))\times\Psi}\\ &-\frac{(n+p)(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q_{n+p-1}^{m}}{(n+(1-\alpha)(p-q))(1-\lambda-\lambda(n+p-q))(n+p-q)^m}B_{n+p-1}\\ =\ &\delta-\frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}B_{n+p-1}\end{aligned}$$

which, in view of definition (3.2), completes the proof of Theorem 7. The proof of other case is similar and so is omitted. □

Remark 1. If the co-analytic part of $g_m(z)$ for $m=0$ is identically zero, then for $p=1, q=\lambda=m=0$ and $p=m=1, q=\lambda=0$ the above neighborhood results were given in Theorem 2.1 and Theorem 2.2 of [13], respectively.

Corollary 4. *If $g_m(z)$ for $m=0$ is in the class $\overline{SH}^{0}_{1,p}(0,0,\alpha)$, then*

$$\overline{SH}^{0}_{1,p}(0,0,\alpha)\subset N^{\delta}_{1,p}(h^{(0)};g_m^{(0)}),$$

where $h(z)$ and $g_m(z)$ are given by (3.2) and

$$\delta=\begin{cases}\dfrac{p(1+p)(1-\alpha)}{1+p(1-\alpha)}-\left(\dfrac{p(1+p)(1+\alpha)}{1+p(1-\alpha)}-1\right)B_p, & \alpha\ge 1-\dfrac{1}{p-q}\\ \dfrac{(1+p)}{1+p(1-\alpha)}-\left(\dfrac{p(1+p)(1+\alpha)}{1+p(1-\alpha)}-1\right)B_p, & \alpha\le 1-\dfrac{1}{p-q}.\end{cases}$$

Corollary 5. *If $g_m(z)$ for $m=0$ is in the class $\overline{SH}^{0}_{1,p}(0,0,0)$, then*

$$\overline{SH}^{0}_{1,p}(0,0,0)\subset N^{\delta}_{1,p}(h^{(0)};\quad g_m^{(0)}),$$

where $h(z)$ and $g_m(z)$ are given by (3.2) and

$$\delta=p.$$

Corollary 6. *If $g_m(z)$ is in the class $\overline{SH}^{1}_{1,1}(0,0,0)$, then*

$$\overline{SH}^{1}_{1,1}(0,0,0)\subset N^{\delta}_{1,1}(h^{(0)};g_m^{(0)}),$$

where $h(z)$ and $g_m(z)$ are given by (3.2) and

$$\delta=\frac{1}{2}(1+B_1).$$

Corollary 7. *If $g_m(z)$ is in the class $\overline{SH}^{m}_{1,1}(0,0,0)$, then*

$$\overline{SH}^{m}_{1,1}(0,0,0)\subset N^{\delta}_{1,1}(h^{(0)};g_m^{(0)}),$$

where $h(z)$ and $g_m(z)$ are given by (3.2) and

$$\delta = \frac{1}{2^m} - \left(\frac{1}{2^m} - 1\right) B_1.$$

Theorem 8. *Let $\lambda < \frac{1}{n+p-q}$, $f_m \in \overline{SH}^m_{n,p}(q,\lambda,\alpha)$ and*

(i) *for $\alpha \geq 1 - \frac{1}{p-q}$,*

$$\delta \leq \frac{(n+p-1)!}{(n+p-1-q)!}\Bigg[p - \Omega \times (1-\alpha)(p-q) + \left(\frac{(n+p-q)!(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q^m_{n+p-1}}{(n+(1-\alpha)(p-q))(1-\lambda-\lambda(n+p-q))(n+p-q)^m(n+p-1)!} - (n+p-1)\right) b_{n+p-1}\Bigg]$$

(ii) *for $\alpha \leq 1 - \frac{1}{p-q}$,*

$$\delta \leq \frac{(n+p-1)!}{(n+p-1-q)!}\Bigg[p - \Omega + \left(\frac{(n+p-q)!(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q^m_{n+p-1}}{(n+(1-\alpha)(p-q))(1-\lambda-\lambda(n+p-q))(n+p-q)^m(n+p-1)!} - (n+p-1)\right) b_{n+p-1}\Bigg]$$

then

$$N^\delta_{n,p}(f^{(q)}_m; g^{(q)}_m) \subset SH^*(n,p),$$

where

$$\Omega = \frac{((1-\lambda)+\lambda(p-q))(n+p-q)!q^m_p}{(n+(1-\alpha)(p-q))(1-\lambda-\lambda(n+p-q))(n+p-q)^m(n+p-1)!}.$$

Proof. Let $\alpha \geq 1 - \frac{1}{p-q}$. Also, suppose that $f_m(z) \in \overline{SH}^m_{n,p}(q,\lambda,\alpha)$ and $g_m(z) \in N^\delta_{n,p}(f^{(q)}_m; g^{(q)}_m)$. We need to show that $g_m(z) \in SH^*(n,p)$. It suffices to show that g_m satisfies the condition (2.1). We have

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} k(A_k+B_k) + (n+p-1)B_{n+p-1} \leq \sum_{k=n+p}^{\infty} k[|a_k-A_k| + |b_k-B_k|] + (n+p-1)|b_{n+p-1}-B_{n+p-1}| \\
&\quad + \sum_{k=n+p}^{\infty} k(a_k+b_k) + (n+p-1)b_{n+p-1} \\
&\leq \frac{(n+p-1-q)!}{(n+p-1)!}\Bigg[\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k[|a_k-A_k| + |b_k-B_k|] \\
&\quad + \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}|b_{n+p-1}-B_{n+p-1}|\Bigg] + (n+p-1)b_{n+p-1} \\
&\quad + \Omega \times \left(\sum_{k=n+p}^{\infty}\left(\frac{(k-q-\alpha(p-q))(1-\lambda+\lambda(k-q))}{(1-\lambda+\lambda(p-q))}\frac{q^m_k}{q^m_p}a_k \right.\right.\\
&\quad \left.\left. + \frac{(k-q+\alpha(p-q))(1-\lambda-\lambda(k-q))}{(1-\lambda+\lambda(p-q))}\frac{q^m_k}{q^m_p}b_k\right)\right) \\
&\leq \frac{(n+p-1-q)!}{(n+p-1)!}\delta + (n+p-1)b_{n+p-1} + \Omega \times \Bigg((1-\alpha)(p-q) \\
&\quad - \frac{(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))}{(1-\lambda+\lambda(p-q))q^m_p}\frac{q^m_{n+p-1}}{q^m_p}b_{n+p-1}\Bigg).
\end{aligned}$$

Now this expression is never greater than p provided that

$$\delta \le \frac{(n+p-1)!}{(n+p-1-q)!}\Bigg[p - \Omega \times (1-\alpha)(p-q) + \left(\frac{(n+p-q)!(n-1+(1+\alpha)(p-q))(1-\lambda-\lambda(n+p-1-q))q_{n+p-1}^m}{(n+(1-\alpha)(p-q))(1-\lambda-\lambda(n+p-q))(n+p-q)^m(n+p-1)!} - (n+p-1)\right) b_{n+p-1}\Bigg].$$

The proof of other case is similar and so is omitted. □

Remark 2. For $m = 1, n = p = 1, q = \lambda = \alpha = 0$, the above neighborhood result was given in [15].

Corollary 8. *If $f_m(z)$ is in the class $\overline{SH}_{1,1}^m(0,0,0)$, then*

$$\overline{SH}_{1,1}^m(0,0,0) \subset N_{1,1}^{\delta}(f_m^{(0)}; \quad g_m^{(0)}),$$

where $f_m(z)$ and $g_m(z)$ are given by (1.5) and (3.2), respectively, and

$$\delta = \left(1 - \frac{1}{2^m}\right)(1-b_1).$$

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