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COMMUTATOR SUBGROUPS OF THE EXTENDED HECKE GROUPS $\overline{H}(\lambda_q)$

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Abstract. Hecke groups $H(\lambda_q)$ are the discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ generated by $S(z) = -(z+\lambda_q)^{-1}$ and $T(z) = -1/z$. The commutator subgroup of $H(\lambda_q)$, denoted by $H'(\lambda_q)$, is studied in [2]. It was shown that $H'(\lambda_q)$ is a free group of rank $q-1$.

Here the extended Hecke groups $\overline{H}(\lambda_q)$, obtained by adjoining $R_1(z) = 1/\bar{z}$ to the generators of $H(\lambda_q)$, are considered. The commutator subgroup of $\overline{H}(\lambda_q)$ is shown to be a free product of two finite cyclic groups. Also it is interesting to note that while in the $H(\lambda_q)$ case, the index of $H'(\lambda_q)$ is changed by q, in the case of $\overline{H}(\lambda_q)$, this number is either 4 for q odd or 8 for q even.

Keywords: Hecke group, extended Hecke group, commutator subgroup

MSC 2000: 11F06, 20H05, 20H10

1. Introduction

In [4], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z+\lambda,$$

where λ is a fixed positive real number. T and U have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

respectively. (In this work we identify each matrix A with $-A$, so that they each represent the same transformation). Let $S = T.U$, i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

E. Hecke showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, where q is an integer $q \geqslant 3$ or $\lambda > 2$ is real. In these two cases $H(\lambda)$ is called a Hecke group. We consider the former case. Then the Hecke group $H(\lambda)$ is the discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by S and U, where

$$U(z) = z + \lambda_q$$

and it has a presentation $H(\lambda) = \langle T, S \mid T^2 = S^q = I \rangle$.

The most important and studied Hecke group is the modular group $H(\lambda_3)$. In this case $\lambda_3 = 2\cos\frac{\pi}{3} = 1$, i.e. all coefficients of the elements of $H(\lambda_3)$ are rational integers. In the literature, the symbols Γ and $\Gamma(1)$ are used to denote the modular group. In this paper we shall use $H(\lambda_3)$ for this purpose. The next two most important Hecke groups are those for $q = 4$ and $q = 6$, in which cases $\lambda_q = \sqrt{2}$ and $\sqrt{3}$, respectively.

The extended modular group $\overline{H}(\lambda_3)$ has a presentation

$$\overline{H}(\lambda_3) = \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_3R_1)^2 = I \rangle$$

where

$$R_1(z) = \frac{1}{\overline{z}}, \quad R_2(z) = \frac{-1}{\overline{z}+1}, \quad R_3(z) = -\overline{z}.$$

The modular group is a subgroup of index 2 in $\overline{H}(\lambda_3)$ (see [3]). It has a presentation

$$H(\lambda_3) = \langle T, S \mid T^2 = S^3 = I \rangle \cong C_2 * C_3,$$

where

$$T = R_3R_1 = R_1R_3, \quad S = R_1R_2.$$

Putting $R = R_1$, we have

$$\overline{H}(\lambda_3) = \langle T, S, R \mid T^2 = S^3 = R^2 = I,\ RT = TR,\ RS = S^{-1}R \rangle .$$

Similarly the extended Hecke group $\overline{H}(\lambda_q)$ has a presentation

$$\overline{H}(\lambda_q) = \langle T, S, R \mid T^2 = S^q = R^2 = I,\ RT = TR,\ RS = S^{-1}R \rangle$$

and Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$.

The commutator subgroup of G is denoted by G' and defined by

$$\langle [g,h] \mid g,h \in G \rangle$$

where $[g,h] = ghg^{-1}h^{-1}$. Since G' is a normal subgroup of G, we can form the factor-group G/G' which is the largest abelian quotient group of G.

In this work we obtain some results concerning commutator subgroups of the extended Hecke group $\overline{H}(\lambda_q)$.

2. Commutator subgroups of the extended Hecke group $\overline{H}(\lambda_q)$

The commutator subgroup of the Hecke group $H(\lambda_q)$ is denoted by $H'(\lambda_q)$. We have

$$T^2 = S^q = I, \quad TS = ST$$

in $H(\lambda_q)/H'(\lambda_q)$. So one can find

$$H(\lambda_q)/H'(\lambda_q) \cong C_2 \times C_q$$

and hence it is isomorphic to C_{2q} if q is odd. Therefore

$$|H(\lambda_q) : H'(\lambda_q)| = 2q.$$

If q is even, $(TS)^q = 1$ while if q is odd, $(TS)^{2q} = 1$. In particular, $H'(\lambda_q)$ is a free group of rank $q-1$ (see [1]).

By [5], the Reidemeister-Schreier method gives the generators of $H'(\lambda_q)$ as

$$a_1 = TSTS^{q-1}, \ a_2 = TS^2TS^{q-2}, \ \ldots, \ a_{q-1} = TS^{q-1}TS.$$

Similarly for the extended Hecke group $\overline{H}(\lambda_q)$ we have

$$T^2 = S^q = R^2 = I, \quad RT = TR, \quad RS = S^{-1}R, \quad RS = SR, \quad TS = ST$$

in $\overline{H}(\lambda_q)/\overline{H}'(\lambda_q)$.

Theorem 1. *Let q be odd, then*
(i) $\overline{H}(\lambda_q)/\overline{H}'(\lambda_q) \cong V_4 \cong C_2 \times C_2$
(ii) $\overline{H}'(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I\rangle \cong C_q * C_q$.

Proof. (i) Since the extended Hecke group $\overline{H}(\lambda_q)$ has a presentation

$$\overline{H}(\lambda_q) = \big\langle T, S, R \mid T^2 = S^q = R^2 = I, \ RT = TR, \ RS = S^{-1}R\big\rangle$$

and

$$\overline{H}(\lambda_q)/\overline{H}'(\lambda_q) = \big\langle T, S, R \mid T^2 = S^q = R^2 = I, \ RT = TR, \ RS = S^{-1}R, \\ RS = SR, \ TS = ST\big\rangle$$

one has $RS = S^{-1}R$ and $RS = SR$, and thus

$$S^{q-2} = S^q = S^2 = I.$$

This shows that $S = I$, as q is odd. Thus

$$\overline{H}(\lambda_q)/\overline{H}'(\lambda_q) = \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle$$

and finally

$$\overline{H}(\lambda_q)/\overline{H}'(\lambda_q) \cong V_4 \cong C_2 \times C_2.$$

(ii) Now we determine the set of generators for $\overline{H}'(\lambda_q)$. We choose a Schreier transversal for $\overline{H}'(\lambda_q)$ as

$$I,\ T,\ R,\ TR.$$

According to the Reidemeister-Schreier method, we can form all possible products

$$\begin{aligned}
&I \cdot T \cdot (T)^{-1} = I, && I \cdot S \cdot (I)^{-1} = S, && I \cdot R \cdot (R)^{-1} = I,\\
&T \cdot T \cdot (I)^{-1} = I, && T \cdot S \cdot (T)^{-1} = TST, && T \cdot R \cdot (TR)^{-1} = I,\\
&R \cdot T \cdot (TR)^{-1} = RTRT, && R \cdot S \cdot (R)^{-1} = RSR, && R \cdot R \cdot (I)^{-1} = I,\\
&TR \cdot T \cdot (R)^{-1} = TRTR, && TR \cdot S \cdot (TR)^{-1} = TRSRT, && TR \cdot R \cdot (T)^{-1} = I.
\end{aligned}$$

Since

$$\begin{aligned}
RTRT &= I,\\
TRTR &= I,\\
RSR &= S^{-1},\\
TRSRT &= TS^{-1}T = (TST)^{-1},
\end{aligned}$$

the generators are S and TST. Thus $\overline{H}'(\lambda_q)$ has a presentation

$$\overline{H}'(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong C_q * C_q.$$

□

Theorem 2. *Let q be even, then*

(i) $\overline{H}(\lambda_q)/\overline{H}'(\lambda_q) \cong C_2 \times C_2 \times C_2$

(ii) $\overline{H}'(\lambda_q) = \langle S^2, TS^2T, TSTS^{q-1} \mid (S^2)^{q/2} = (TS^2T)^{q/2} = (TSTS^{q-1})^{\infty} = I \rangle$.

Proof. (i) If the representations of $\overline{H}(\lambda_q)$ and $\overline{H}(\lambda_q)/\overline{H}'(\lambda_q)$ are considered, we obtain $S^2 = I$ as $RS = S^{-1}R$ and $RS = SR$, $S^{q-2} = S^q = S^2 = I$ as q is odd. Therefore

$$\overline{H}(\lambda_q)/\overline{H}'(\lambda_q) = \langle T, S, R \mid T^2 = S^2 = R^2 = (RT)^2 = (RS)^2 = (TS)^2 = I \rangle$$

and so

$$\overline{H}(\lambda_q)/\overline{H}'(\lambda_q) \cong C_2 \times C_2 \times C_2.$$

(ii) Again we choose a Schreier transversal for $\overline{H}'(\lambda_q)$ as

$$I,\ T,\ R,\ S,\ TR,\ SR,\ TS,\ TSR.$$

Hence, all possible products are

$$\begin{aligned}
I \cdot T \cdot (T)^{-1} &= I, & TR \cdot T \cdot (R)^{-1} &= TRTR,\\
T \cdot T \cdot (I)^{-1} &= I, & SR \cdot T \cdot (TSR)^{-1} &= SRTRS^{-1}T,\\
R \cdot T \cdot (TR)^{-1} &= RTRT, & TS \cdot T \cdot (S)^{-1} &= TSTS^{-1},\\
S \cdot T \cdot (TS)^{-1} &= STS^{-1}T, & TSR \cdot T \cdot (SR)^{-1} &= TSRTRS^{-1},\\
I \cdot S \cdot (S)^{-1} &= I, & TR \cdot S \cdot (TSR)^{-1} &= TRSRS^{-1}T,\\
T \cdot S \cdot (TS)^{-1} &= I, & SR \cdot S \cdot (R)^{-1} &= SRSR,\\
R \cdot S \cdot (SR)^{-1} &= RSRS^{-1}, & TS \cdot S \cdot (T)^{-1} &= TS^2T,\\
S \cdot S \cdot (I)^{-1} &= S^2, & TSR \cdot S \cdot (TR)^{-1} &= TSRSRT,\\
I \cdot R \cdot (R)^{-1} &= I, & TR \cdot R \cdot (T)^{-1} &= I,\\
T \cdot R \cdot (TR)^{-1} &= I, & SR \cdot R \cdot (S)^{-1} &= I,\\
R \cdot R \cdot (I)^{-1} &= I, & TS \cdot R \cdot (TSR)^{-1} &= I,\\
S \cdot R \cdot (SR)^{-1} &= I, & TSR \cdot R \cdot (TS)^{-1} &= I.
\end{aligned}$$

Since $(STS^{-1}T)^{-1} = TSTS^{-1}$, $(TRTR)^{-1} = RTRT = I$, $(RSRS^{-1}) = (S^2)^{-1}$, $SRSR = I$, $SRTRS^{-1}T)^{-1} = TSRTRS^{-1} = TSTS^{-1}$, $TRSRS^{-1}T = (TS^2T)^{-1}$, $TSRSRT = I$, the generators of $\overline{H}'(\lambda_q)$ are S^2, TS^2T, $TSTS^{q-1}$. Thus $\overline{H}'(\lambda_q)$ has a presentation

$$\overline{H}'(\lambda_q) = \left\langle S^2 \right\rangle * \left\langle TS^2T \right\rangle * \left\langle TSTS^{q-1} \right\rangle.$$

Example 1. Let $q = 3$. Then $\overline{H}(\lambda_3)$ is the extended modular group. In this case

$$\overline{H}(\lambda_3)/\overline{H}'(\lambda_3) = \left\langle T, R \mid T^2 = R^2 = (TR)^2 = I \right\rangle$$

and a Schreier transversal is

$$I,\ T,\ R,\ TR.$$

Hence,

$$\begin{aligned}
I \cdot T \cdot (T)^{-1} &= I, & I \cdot S \cdot (I)^{-1} &= S, & I \cdot R \cdot (R)^{-1} &= I,\\
T \cdot T \cdot (I)^{-1} &= I, & T \cdot S \cdot (T)^{-1} &= TST, & T \cdot R \cdot (TR)^{-1} &= I,\\
R \cdot T \cdot (TR)^{-1} &= RTRT, & R \cdot S \cdot (R)^{-1} &= RSR, & R \cdot R \cdot (I)^{-1} &= I,\\
TR \cdot T \cdot (R)^{-1} &= TRTR, & TR \cdot S \cdot (TR)^{-1} &= TRSRT, & TR \cdot R \cdot (T)^{-1} &= I
\end{aligned}$$

and since $RTRT = I$, $TRTR = I$, $RSR = S^{-1}$, $TRSRT = TS^{-1}T = (TST)^{-1}$, the generators of $\overline{H}(\lambda_3)$ are S and TST. Thus $\overline{H}'(\lambda_3)$ has a presentation

$$\overline{H}'(\lambda_3) = \left\langle S, TST \mid S^3 = (TST)^3 = I \right\rangle \cong C_3 * C_3.$$

Notice that this result coincides with the ones given in [5] for the extended modular group.

Example 2. Let $q = 6$. Then $\overline{H}(\lambda_6)$ and $\overline{H}(\lambda_6)/\overline{H}'(\lambda_6)$ have presentations

$$\overline{H}(\lambda_6) = \left\langle T, S, R \mid T^2 = S^6 = R^2 = I,\ TR = RT,\ RS = S^{-1}R \right\rangle$$

and

$$\overline{H}(\lambda_6)/\overline{H}'(\lambda_6) = \langle T, S, R \mid T^2 = S^6 = R^2 = I,\ RT = TR,\ RS = S^{-1}R,\\ RS = SR,\ TS = ST \rangle .$$

Since $RS = S^{-1}R$ and $RS = SR$, $S^{-1} = S^5$ and so $S^4 = S^6 = I$, $S^2 = I$. Hence

$$\overline{H}(\lambda_6)/\overline{H}'(\lambda_6) = \left\langle T, S, R \mid T^2 = S^2 = R^2 = (RT)^2 = (RS)^2 = (TS)^2 = I \right\rangle .$$

We can choose a Schreier transversal as

$$I,\ T,\ R,\ S,\ TR,\ SR,\ TS,\ TSR.$$

In this case all the possibilities are

$$\begin{aligned}
I \cdot T \cdot (T)^{-1} &= I, & TR \cdot T \cdot (R)^{-1} &= TRTR,\\
T \cdot T \cdot (I)^{-1} &= I, & SR \cdot T \cdot (TSR)^{-1} &= SRTRS^5T,\\
R \cdot T \cdot (TR)^{-1} &= RTRT, & TS \cdot T \cdot (S)^{-1} &= TSTS^5,\\
S \cdot T \cdot (TS)^{-1} &= STS^5T, & TSR \cdot T \cdot (SR)^{-1} &= TSRTRS^5,\\
I \cdot S \cdot (S)^{-1} &= I, & TR \cdot S \cdot (TSR)^{-1} &= TRSRS^5T,\\
T \cdot S \cdot (TS)^{-1} &= I, & SR \cdot S \cdot (R)^{-1} &= SRSR,\\
R \cdot S \cdot (SR)^{-1} &= RSRS^5, & TS \cdot S \cdot (T)^{-1} &= TS^2T,\\
S \cdot S \cdot (I)^{-1} &= S^2, & TSR \cdot S \cdot (TR)^{-1} &= TSRSRT,\\
I \cdot R \cdot (R)^{-1} &= I, & TR \cdot R \cdot (T)^{-1} &= I,\\
T \cdot R \cdot (TR)^{-1} &= I, & SR \cdot R \cdot (S)^{-1} &= I,\\
R \cdot R \cdot (I)^{-1} &= I, & TS \cdot R \cdot (TSR)^{-1} &= I,\\
S \cdot R \cdot (SR)^{-1} &= I, & TSR \cdot R \cdot (TS)^{-1} &= I.
\end{aligned}$$

Since $(STS^5T)^{-1} = TSTS^5$, $(TRTR)^{-1} = RTRT = I$, $(RSRS^5) = (S^2)^{-1}$, $SRSR = I$, $(SRTRS^5T)^{-1} = TSRTRS^5 = TSTS^5$, $TRSRS^5T = (TS^2T)^{-1}$, $TSRSRT = I$, the generators of $\overline{H}'(\lambda_q)$ are S^2, TS^2T, $TSTS^5$. Thus $\overline{H}'(\lambda_6)$ has a presentation

$$\overline{H}'(\lambda_6) = \left\langle S^2 \right\rangle * \left\langle TS^2T \right\rangle * \left\langle TSTS^5 \right\rangle .$$

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