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TENSOR PRODUCT SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. Tensor product immersions of a given Riemannian manifold was initiated by B.-Y. Chen. In the present article we study the tensor product surfaces of two Euclidean plane curves. We show that a tensor product surface M of a plane circle c_1 centered at origin with an Euclidean planar curve c_2 has harmonic Gauss map if and only if M is a part of a plane. Further, we give necessary and sufficient conditions for a tensor product surface M of a plane circle c_1 centered at origin with an Euclidean planar curve c_2 to have pointwise 1-type Gauss map.

1. Introduction

Since the late 1970's, the study of submanifolds of Euclidean space or pseudo-Euclidean space with the notion of finite type immersion has been extensively carried out. An isometric immersion $x : M \to \mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ is said to be of finite type if x identified with the position vector field of M in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is; $x = x_0 + \sum_{i=1}^{k} x_i$ where x_0 is a constant map $x_1, x_2, \ldots, x_k$ non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different, then M is said to be of k-type. Similarly, a smooth map ϕ of an n-dimensional Riemannian manifold M of $\mathbb{E}^m$ is said to be of finite type if ϕ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of Δ ([5], [6]). Granted, this notion of finite type immersion is naturally extended to the Gauss map G on M in Euclidean space ([9]). Thus, if a submanifold M of Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C ([1], [2], [3], [14]). However, the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and a

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right cone in Euclidean 3-space $\mathbb{E}^3$ take a somewhat different form; namely, $\Delta G = f(G+C)$ for some non-constant function f and some constant vector C. Therefore, it is worth studying the class of solution surfaces satisfying such an equation. A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to have pointwise 1-type Gauss map if its Gauss map G satisfies

$$\Delta G = f(G + C) \tag{1}$$

for some non-zero smooth function f on M and a constant vector C. A pointwise 1-type Gauss map is called proper if the function f defined by (1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([8], [10], [15], [16]). In [10], two of the present authors characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with B.-Y. Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincides with surfaces of revolution with constant mean curvature. Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map [8]. In [18] D. W. Yoon study with Vraneanu rotation surfaces in Euclidean 4-space $\mathbb{E}^4$. He obtain the complete classification theorems for the flat Vraneanu rotation surfaces with 1-type Gauss map and an equation in terms of the mean curvature vector. For more detail see also [17].

The study of tensor product immersion of two immersions of a given Riemannian manifold was introduced by B.-Y. Chen (See, [7]). Further, product immersions of two plane curves were studied in [13] as a surface in $\mathbb{E}^4$. In this article we investigate a tensor product surface with pointwise 1- type Gauss map in Euclidean 4-space $\mathbb{E}^4$. First, we consider the tensor product immersions with harmonic Gauss map. Further we investigate tensor product immersions of two plane curves with pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$.

2. Preliminaries

In the present section we recall definitions and results of [4]. Let $x : M \to \mathbb{E}^m$ be an immersion from an n-dimensional connected Riemannian manifold M into an m-dimensional Euclidean space $\mathbb{E}^m$. We denote by g the metric tensor of $\mathbb{E}^m$ as well as the induced metric on M. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Then the Gaussian and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{3}$$

where X, Y are vector fields tangent to M and ξ normal to M. Moreover, h is the second fundamental form, D is the linear connection induced in the normal bundle $T^\perp M$, called normal connection and A_ξ the shape operator in the direction of ξ that is related with h by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle.$$

If we define a covariant differentiation $\overline{\nabla}h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of M by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. Then we have the Codazzi equation

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z). \tag{4}$$

We denote R, the curvature tensor associated with ∇;

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5}$$

The equations Gauss and Ricci are given respectively by

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z)h(Y,W)\rangle, \tag{6}$$

$$\langle [A_\xi, A\eta]X, Y\rangle = 0 \tag{7}$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M.

For an n-dimensional submanifold M in $\mathbb{E}^m$. The mean curvature vector $\overrightarrow{H}$ is given by

$$\overrightarrow{H} = \frac{1}{n}\mathrm{trace} h.$$

A submanifold M is said to be minimal (respectively, totally geodesic) if $\overrightarrow{H} \equiv 0$ (respectively, $h \equiv 0$).

Let us now define the Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n \mathbb{E}^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \binom{m}{n}$. Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an adapted local orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, \ldots, e_n$, are tangent to M and $e_{n+1}, \ldots, e_{n+2}, \ldots, e_m$ normal to M. The map $G : M \to G(n,m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ is called the Gauss map of M that is a smooth map which carries a point p in M into the oriented n-plane in $\mathbb{E}^m$ obtained from the parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

For any real function f on M the Laplacian of f is defined by

$$\Delta f = -\sum_i (\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} f - \widetilde{\nabla}_{\nabla_{e_i} e_i} f). \tag{8}$$

3. Tensor product surfaces with finite type Gauss map

In the following sections, we will consider the tensor product immersions, actually surfaces in $\mathbb{E}^4$, which are obtained from two Euclidean plane curves. Let $c_1 : \mathbb{R} \to \mathbb{E}^2$ and $c_2 : \mathbb{R} \to \mathbb{E}^2$ be two Euclidean curves. Put $c_1(t) = (\gamma(t), \delta(t))$ and $c_2(s) = (\alpha(s), \beta(s))$. Then their tensor product surface is given by

$$f = c_1 \otimes c_2 : \mathbb{R}^2 \to \mathbb{E}^4,$$

$$f(t,s) = (\alpha(s)\gamma(t), \beta(s)\gamma(t), \alpha(s)\delta(t), \beta(s)\delta(t)) \tag{9}$$

(See [11] and [13]). If we take c_1 as a unit plane circle centered at 0 and $c_2(s) = (\alpha(s), \beta(s))$ is a unit speed Euclidean plane curve, then the surface patch becomes

$$M : \quad f(t,s) = (\alpha(s)\cos t, \beta(s)\cos t, \alpha(s)\sin t, \beta(s)\sin t). \tag{10}$$

An orthonormal frame tangent to M is given by

$$e_1 = \frac{1}{\|c_2\|}\frac{\partial f}{\partial t}, \tag{11}$$

$$e_2 = \frac{\partial f}{\partial s}. \tag{12}$$

The normal space of M is spanned by

$$n_1 = (-\beta'(s)\cos t, \beta'(s)\cot s, \alpha'(s)\sin t, -\alpha'(s)\ \sin t), \tag{13}$$

$$n_2 = \frac{1}{\|c_2\|}(-\beta(s)\sin t, \beta(s)\sin t, \alpha(s)\cos t, -\alpha(s)\cos t). \tag{14}$$

By covariant differentiation with respect to e_1 and e_2 a straightforward calculation gives

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= -a(s)e_2 + b(s)n_1, \\ \tilde{\nabla}_{e_2} e_2 &= c(s)n_1, \\ \tilde{\nabla}_{e_2} e_1 &= -b(s)n_2, \\ \tilde{\nabla}_{e_1} e_2 &= a(s)e_1 - b(s)n_2, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \tilde{\nabla}_{e_1} n_1 &= -b(s)e_1 - a(s)n_2, \\ \tilde{\nabla}_{e_1} n_2 &= b(s)e_2 + a(s)n_1, \\ \tilde{\nabla}_{e_2} n_1 &= -c(s)e_2, \\ \tilde{\nabla}_{e_2} n_2 &= -b(s)e_1, \end{aligned} \tag{16}$$

where

$$a(s) = \frac{\alpha(s)\alpha'(s) + \beta(s)\beta'(s)}{\|c_2(s)\|^2}, \tag{17}$$

$$b(s) = \frac{\alpha(s)\beta'(s) - \beta(s)\alpha'(s)}{\|c_2(s)\|^2}, \tag{18}$$

$$c(s) = \alpha'(s)\beta''(s) - \alpha''(s)\beta'(s). \tag{19}$$

are the differentiable functions.

By the use of (16) with (3) we get the following result.

Lemma 3.1. *Let $f = c_1 \otimes c_2$ be a tensor product immersion of a plane circle c_1 centered at the origin with any Euclidean planar curve $c_2(s) = (\alpha(s), \beta(s))$. Then*

$$A_{n_1} = \begin{bmatrix} b(s) & 0 \\ 0 & c(s) \end{bmatrix}, \; A_{n_2} = \begin{bmatrix} 0 & -b(s) \\ -b(s) & 0 \end{bmatrix}. \tag{20}$$

By using (8), (15), (16) and straight-forward computation the Laplacian ΔG of the Gauss map can be expressed as

$$\begin{aligned} -\Delta G &= (-2a(s)b(s) + c'(s) + a(s)c(s))\, e_1 \wedge e_3 \\ &\quad + (2a(s)b(s) + b'(s) + a(s)b(s)) e_2 \wedge e_4 \\ &\quad + \left(3b^2(s) + c^2(s)\right) e_2 \wedge e_1 + \left(2b(s)c(s) - 2b^2(s)\right) e_3 \wedge e_4. \end{aligned} \tag{21}$$

First, we suppose that the Gauss map of M is harmonic, i.e., $\Delta G = \overrightarrow{0}$. From (21) we get

$$\begin{aligned} 3b^2(s) + c^2(s) &= 0, \\ b(s)c(s) - b^2(s) &= 0, \\ -2a(s)b(s) + c'(s) + a(s)c(s) &= 0, \\ 2a(s)b(s) + b'(s) + a(s)b(s) &= 0. \end{aligned} \tag{22}$$

Then, the first equation of (22) implies that $b = 0$ and $c = 0$. So, by (20), M is a totally geodesic surface in $\mathbb{E}^4$.

Thus we have:

Theorem 3.2. *Let M be a tensor product surface of a plane circle c_1 centered at the origin with a Euclidean planar curve $c_2(s) = (\alpha(s), \beta(s))$. If the Gauss map of M is harmonic, then it is a part of a plane.*

Now, we suppose that the rotation surface M is of pointwise 1-type Gauss map in $\mathbb{E}^4$. From (1) and (21)

$$\begin{aligned} f + f\langle C, e_1 \wedge e_2\rangle &= -3b^2(s) - c^2(s), \\ f\langle C, e_1 \wedge e_3\rangle &= -2a(s)b(s) + c'(s) + a(s)c(s), \\ f\langle C, e_2 \wedge e_4\rangle &= 2a(s)b(s) + b'(s) + a(s)b(s), \\ f\langle C, e_3 \wedge e_4\rangle &= 2b(s)c(s) - 2b^2(s), \end{aligned} \tag{23}$$

where f is a smooth non-zero function. Then we obtain from (21)

$$\begin{aligned} f\langle C, e_1 \wedge e_4\rangle &= 0, \\ f\langle C, e_2 \wedge e_3\rangle &= 0. \end{aligned} \tag{24}$$

Further, by using the equations of Gauss, Codazzi and Ricci after some computation we get

$$a'(s) + a^2(s) = b^2(s) - b(s)c(s), \tag{25}$$

$$b'(s) = -2a(s)b(s) + a(s)c(s), \tag{26}$$

and

$$b(s)\left(b(s) - c(s)\right) = 0, \tag{27}$$

respectively.

Consider the open subset $U = \{s \in \mathrm{dom} c_2 \mid b(s) \neq c(s)\}$. Suppose $U \neq \emptyset$. Then, $b(s) = 0$ on U by (27). (26) with it implies $a(s)c(s) = 0$. If $a(s_0) \neq 0$ for some $s_0 \in U$, then $c(s_0) = 0$, a contradiction. Thus, $a(s) = 0$ on U. Hence, (17) and (18) show that $c_2(s) = (\alpha(s), \beta(s)$ is a constant vector on U, a contradiction. Therefore, $b(s) = c(s)$ for all s. Hence, from (25) one can get a Bernoulli differential equation

$$a'(s) + a^2(s) = 0.$$

Thus, one can have a trivial solution

$$a(s) \equiv 0$$

or a non-trivial solution

$$a(s) = \frac{1}{s + s_0} \tag{28}$$

for some constant s_0.

Suppose $a \equiv 0$. By (26), b is a constant and so is c. By (23) with $b(s) = c(s) = \text{const.}$, the constant vector C reduces to

$$C = \langle C, G\rangle G$$

and thus $\langle C, G\rangle G$ is constant. Therefore, the Gauss map G is eventually a constant vector. In this case, M is part of a plane.

Let us consider the case that a has a non-trivial solution. Combining (28) with (17), we obtain a differential equation

$$\frac{(\alpha^2(s) + \beta^2(s))'}{2(\alpha^2(s) + \beta^2(s))} = \frac{1}{s + s_0}$$

which has a solution

$$\alpha^2(s) + \beta^2(s) = \mu(s + s_0)^2$$

for some non-zero constant μ.

Since $c_2(s) = (\alpha(s), \beta(s))$ is of unit speed, we may put

$$\alpha'(s) = \cos\theta(s), \ \beta'(s) = \sin\theta(s) \tag{29}$$

for some function $\theta(s)$ and using (19) we get

$$\begin{aligned} c(s) &= \alpha'(s)\beta''(s) - \alpha''(s)\beta'(s) \\ &= \theta'(s). \end{aligned}$$

Furthermore, substituting $c(s) = b(s)$ into (26) and using (28) we obtain

$$b'(s) = -\frac{b(s)}{s + s_0},$$

which has the solution

$$b(s) = \frac{\lambda}{s + s_0}, \ \lambda = \text{const.} \tag{30}$$

Combining (30), (29) and using $c(s) = b(s)$, we get

$$\theta(s) = \lambda \ln |s + s_0| \, .$$

So, substituting this into (29) we get

$$\begin{aligned} \alpha(s) &= \int \cos(\lambda \ln |s + \mu|) ds, \\ \beta(s) &= \int \sin(\lambda \ln |s + \mu|) ds, \end{aligned} \tag{31}$$

The converse also holds.

Thus, summing up the following theorem is proved.

Theorem 3.3. *Let M be a tensor product surface of a plane circle c_1 centered at the origin with a Euclidean planar curve $c_2(s) = (\alpha(s), \beta(s))$. Then M has pointwise 1-type Gauss map if and only if M is either totally geodesic or parameterized by*

$$\begin{aligned} \alpha(s) &= \int \cos(\lambda \ln |s + \mu|) ds, \\ \beta(s) &= \int \sin(\lambda \ln |s + \mu|) ds. \end{aligned}$$

Remark. Part of plane can be considered as a surface of a Euclidean space with pointwise 1-type Gauss map of the second kind.

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