Pure Bending of Elliptical Ring Sector with Cross Section of Multi-Connected Region Composed of Confocal Ellipses*

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In this study, internal stresses of an elliptical ring sector with the cross section of a multi connected region composed of two confocal ellipses, subjected to pure bending are analyzed. Göhner's method is used for analysis and therefore, some difficulties caused by elliptical coordinates are eliminated. The analysis is limited to determining the first correction to the initial stress state for pure bending of an elliptical ring sector with the cross section of two confocal ellipses.

Key Words: Bending, Elliptical, Ring Sector, Multi Connected, Region, Confocal, Ellipses

1. Introduction

In this study, we consider the symmetric elastic stress distribution of an elastic toroidal ring sector subjected to pure bending under the effect of moments applied to its free ends. The cross section of the ring sector consists of two confocal ellipses. Such an investigation will enable the study of a toroidal ring sector whose cross-section consists of two concentric circles.

The present work extends the works of Lang[1][2] who derived analytical expressions for the pure bending of an elliptical ring sector in forms particularly useful for ring sectors with small amounts of ξ/R, where ξ and R are the coordinate parameter and the radius of the ring, respectively. Lang[1] also provided some formulas for a circular ring sector as a special case and obtained the same results for the circular ring sector given in Ref. (3). Stress field in a circular ring sector was studied by Göhner[4] and the other authors[5]-[9]. Göhner obtained analytical results for a circular toroidal ring sector by using the method of successive approximations. Lang's work is a generalization of circular sections which includes the elliptical sections.

At this point, we are concerned with the stress distribution of a toroidal ring sector with a cross section composed of two concentric circles. Such an analysis enables us to examine elbow pipe elements. However, it is more convenient to formulate the problem by considering an elliptic ring sector with a cross section consisting of two confocal ellipses since it will provide a generalization, and the cross section of the ring sector may have nearly elliptical shape near the neck owing to the ovalization under the effect of bending moments. For this aim, in this work, we take into account the elliptical ring sector with the section consisting of two confocal ellipses. There are two method used for the analysis: the method of toroidal elasticity[2] and the method of Göhner[4]. In view of the simplicity, we use the method of Göhner to avoid the use of elliptical coordinates.

The analysis is limited to determining the first correction to the initial stress state for pure bending

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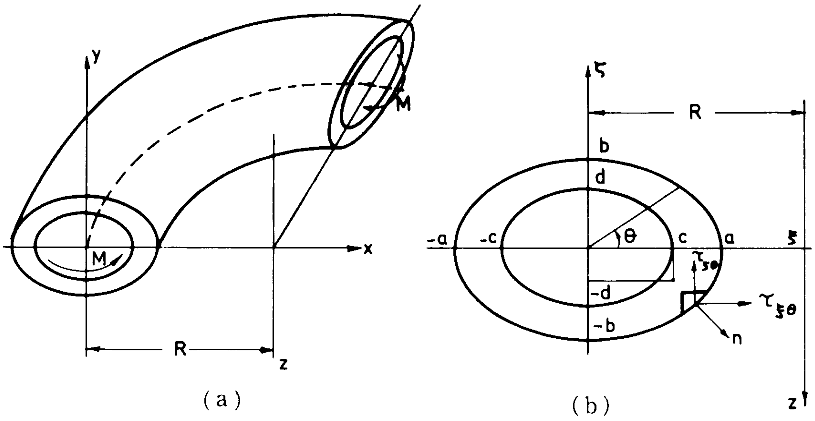


Fig. 1 Elliptic ring sector subjected to pure bending

of an elliptical ring sector.

2. Analysis

Consider an elliptical ring sector subjected to bending moments as shown in Fig. 1. Coordinates are chosen as in Fig. 1(b). We use the cylindrical coordinate system for the analysis. Notations for stresses in this case are σ_r, σ_θ, σ_z, $\tau_{r\theta}$, τ_{rz} and $\tau_{\theta z}$. Displacements corresponding to radial, circumferential and z directions will be given as u, v and w, respectively. However, if a couple of moments are applied to the free ends of the ring in the ring plane (see, Fig. 1), these moments produce symmetric deformations and the stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ vanish. The remaining stress components must satisfy the equilibrium equations

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0$$
$$\frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0 \quad (1)$$

and the corresponding compatibility equations

$$\nabla^2\sigma_r-\frac{2}{r^2}(\sigma_r-\sigma_\theta)+\frac{1}{1+\nu}\frac{\partial^2\oplus}{\partial r^2}=0$$
$$\nabla^2\sigma_\theta+\frac{2}{r^2}(\sigma_r-\sigma_\theta)+\frac{1}{(1+\nu)}\frac{1}{r}\frac{\partial\oplus}{\partial r}=0$$
$$\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\oplus}{\partial z^2}=0 \quad (2)$$
$$\nabla^2\tau_{rz}-\frac{1}{r^2}\tau_{rz}+\frac{1}{(1+\nu)}\frac{\partial^2\oplus}{\partial r\partial z}=0,$$

where ∇^2 and $\oplus$ are $\partial^2/\partial x^2+\partial^2/\partial y^2$ and $\sigma_r+\sigma_\theta+\sigma_z$, respectively. It must be remembered that the body forces are neglected in this formulation.

Now, introducing the transformations

$$r=R-\xi,\ \zeta=z, \quad (3)$$

Eqs. (1) and (2) can be rewritten as

$$\frac{\partial\sigma_\xi}{\partial\xi}-\frac{\partial\tau_{\xi\zeta}}{\partial\zeta}-\frac{\sigma_\xi-\sigma_\theta}{R-\xi}=0$$
$$\frac{\partial\tau_{\xi\zeta}}{\partial\xi}-\frac{\partial\sigma_\zeta}{\partial\zeta}-\frac{\tau_{\xi\zeta}}{R-\xi}=0 \quad (4)$$

and

$$\frac{\partial^2\sigma_\xi}{\partial\xi^2}+\frac{\partial^2\sigma_\xi}{\partial\zeta^2}-\frac{1}{R-\xi}\frac{\partial\sigma_\xi}{\partial\xi}-\frac{2}{(R-\xi)^2}(\sigma_\xi-\sigma_\zeta)$$
$$+\frac{1}{(1+\nu)}\frac{\partial^2\oplus}{\partial\xi^2}=0$$
$$\frac{\partial^2\sigma_\theta}{\partial\xi^2}+\frac{\partial^2\sigma_\theta}{\partial\zeta^2}-\frac{1}{R-\xi}\frac{\partial\sigma_\theta}{\partial\xi}+\frac{2}{(R-\xi)^2}(\sigma_\xi-\sigma_\theta)$$
$$-\frac{1}{(1+\nu)}\frac{1}{(R-\xi)}\frac{\partial\oplus}{\partial\xi}=0$$
$$\frac{\partial^2\sigma_\zeta}{\partial\xi^2}+\frac{\partial^2\sigma_\zeta}{\partial\zeta^2}-\frac{1}{R-\xi}\frac{\partial\sigma_\zeta}{\partial\xi}+\frac{1}{1+\nu}\frac{\partial^2\oplus}{\partial\zeta^2}=0$$
$$\frac{\partial^2\tau_{\xi\zeta}}{\partial\xi^2}+\frac{\partial^2\tau_{\xi\zeta}}{\partial\zeta^2}-\frac{1}{R-\xi}\frac{\partial\tau_{\xi\zeta}}{\partial\xi}+\frac{1}{(R-\xi)^2}\tau_{\xi\zeta}$$
$$+\frac{1}{1+\nu}\frac{\partial^2\oplus}{\partial\xi\partial\zeta}=0. \quad (5)$$

using the transformation equations (3), where ν is Poisson's ratio. As a first approximation, we assume that the initial stress state in the toroidal ring sector is identical with the stress state arising in the pure bending of prismatic bars:

$$(\sigma_\xi)_o=(\sigma_\zeta)_o=(\tau_{\xi\zeta})_o=0,\ (\sigma_\theta)_o=-cE\xi, \quad (6)$$

where c is a constant to be determined. In order to obtain a second approximation, we assume that ξ is small compared to R. Thus, ξ/R can be taken as zero. Under this assumption, Eqs. (4) and (5) give

$$\frac{\partial(\sigma_\xi)_1}{\partial\xi}-\frac{\partial(\tau_{\xi\zeta})_1}{\partial\zeta}-\frac{cE\xi}{R}=0$$
$$\frac{\partial(\tau_{\xi\zeta})_1}{\partial\xi}-\frac{\partial(\sigma_\zeta)_1}{\partial\zeta}=0 \quad (7)$$

and

$$\Delta(\sigma_\xi)_1+\frac{1}{1+\nu}\frac{\partial^2\oplus_1}{\partial\xi^2}=0$$
$$\Delta(\sigma_\zeta)_1+\frac{1}{1+\nu}\frac{\partial^2\oplus_1}{\partial\zeta^2}=0$$
$$\Delta(\sigma_\theta)_1+\frac{2+\nu}{1+\nu}\frac{cE}{R}=0 \quad (8)$$
$$\Delta(\tau_{\xi\zeta})_1-\frac{1}{1+\nu}\frac{\partial^2\oplus_1}{\partial\xi\partial\zeta}=0.$$

Next we introduce a stress function ϕ such that

$$\sigma_\xi=\frac{cE}{2b^2R}(b^2\xi^2+a^2\zeta^2-a^2b^2)+\frac{\partial^2\phi}{\partial\zeta^2} \quad (9)$$

$$\sigma_\zeta=\frac{\partial^2\phi}{\partial\xi^2},\ \tau_{\xi\zeta}=\frac{\partial^2\phi}{\partial\xi\partial\zeta},$$

where a and b are the principal axes of the outer ellipse. Such a selection of stress field satisfies the equilibrium equations. We note here that the stress field proposed is chosen such that no coupling between the dimensions of two ellipses occurs. Otherwise, as will be seen in the following, there would have been no possibility of reducing the general expressions to the case of circular sections.

The boundary condition that the stress function ϕ must satisfy is obtained from the boundary equations[3]

$$\bar{X}=l\sigma_\xi+m\tau_{\xi\zeta}$$
$$\bar{Y}=m\sigma_\zeta+l\tau_{\xi\zeta}, \quad (10)$$

where X, Y and l, m are the surface tractions and cosine directions, respectively. Since there exists no surface traction on the surface and $l=d\zeta/ds$ and $m=-d\xi/ds$, we can write

$$\frac{d}{ds}\left(\frac{\partial\phi}{\partial\zeta}\right)=0,\ \frac{d}{ds}\left(\frac{\partial\phi}{\partial\xi}\right)=0, \quad (11)$$

where s is a line element on the surface. Thus, $\partial\phi/\partial\zeta$ and $\partial\phi/\partial\xi$ are constants on the surface and $\phi=0$ and $d\phi/dn=0$ on the force-free surface. We keep in mind here that the stress function ϕ must also be zero on the inner surface defined by the boundary equation $x^2/c^2+y^2/d^2=1$, where c and d are the principal axes of the inner ellipse.

The sum of the first three compatibility equations is

$$\varDelta\oplus=-\frac{cE}{R}. \quad (12)$$

Subtracting

$$\varDelta\sigma_\theta=-\left(\frac{2+\nu}{1+\nu}\right)\frac{cE}{R} \quad (13)$$

from Eq. (12), we have

$$\varDelta(\sigma_\xi+\sigma_\zeta)=\frac{1}{1+\nu}\frac{cE}{R}. \quad (14)$$

On the other hand, since

$$\sigma_\xi+\sigma_\zeta=\frac{cE}{2b^2R}(b^2\xi^2+a^2\zeta^2-b^2a^2)+\varDelta\phi, \quad (15)$$

it is found that

$$\varDelta(\sigma_\xi+\sigma_\zeta)=\varDelta\varDelta\phi+\frac{cE}{b^2R}(a^2+b^2). \quad (16)$$

Thus, from Eqs. (14) and (16), we obtain

$$\varDelta\varDelta\phi=\frac{cE(-\nu b^2-(1+\nu)a^2)}{b^2R(1+\nu)}. \quad (17)$$

The stress function ϕ must be chosen such that it becomes zero on the outer and inner surfaces, and this can be achieved by taking

$$\phi=\frac{cEA_o}{64R}[(b^2\xi^2+a^2\zeta^2-a^2b^2)(d^2\xi^2+c^2\zeta^2-c^2d^2)], \quad (18)$$

where A_o is a constant to be determined.

Using Eq. (18), we obtain

$$\varDelta\varDelta\phi=\frac{cEA_o}{8R}[3(b^2d^2+a^2c^2)+(b^2c^2+a^2d^2)]. \quad (19)$$

The comparison of Eqs. (17) and (19) leads to

$$A_o=\frac{-8[\nu b^2+(1+\nu)a^2]}{(1+\nu)b^2[3(b^2d^2+a^2c^2)+(b^2c^2+a^2d^2)]}. \quad (20)$$

We take the stress σ_θ in the form of[1]

$$\sigma_\theta=\frac{cE}{R}[c_o+c_1\xi^2+c_2\zeta^2], \quad (21)$$

where c_0, c_1 and c_2 are constants to be determined. c_0 can be determined from the fact that there is no resultant force on the cross section of the ring:

$$N=0=\iint\sigma_\theta d\xi d\zeta=\frac{cE}{R}\iint(c_o+c_1\xi^2+c_2\zeta^2)d\xi d\zeta. \quad (22)$$

This integral must be integrated over the section surrounded by two ellipses (see, Fig. 1(b)). The result is

$$c_o=\frac{-1}{4(ab-cd)}[c_1(ba^3-dc^3)+c_2(ab^3-cd^3)]. \quad (23)$$

c_1 and c_2 can be determined from the compatibility equations. To this end, we again write the stress fields σ_ξ, σ_ζ, σ_θ and $\tau_{r\theta}$ using Eq. (18):

$$\sigma_\xi=\frac{cE}{2b^2R}(b^2\xi^2+a^2\zeta^2-a^2b^2)+\frac{cEA_o}{32R}[\mu_1^2\xi^2+\mu_2^2\zeta^2-\mu_3^2]$$
$$\sigma_\zeta=\frac{cEA_o}{32R}[\mu_4^2\xi^2+\mu_1^2\zeta^2-\mu_5^2]$$
$$\tau_{\xi\zeta}=\frac{cEA_o}{16R}[b^2c^2+a^2d^2]\zeta\xi$$
$$\sigma_\theta=\frac{cE}{4(ab-cd)R}\Big[c_1(dc^3-ba^3)+4(ab-cd)\xi^2]+c_2[(cd^3-ab^3)+4(ab-cd)\zeta^2]\Big], \quad (24)$$

where μ_1^2, μ_2^2, μ_3^2, μ_4^2 and μ_5^2 are given by

$$\mu_1^2=a^2d^2+b^2c^2,\ \mu_2^2=6a^2c^2,\ \mu_3^2=a^2c^2d^2+a^2b^2c^2$$
$$\mu_4^2=6b^2d^2,\ \mu_5^2=c^2b^2d^2+a^2b^2d^2. \quad (25)$$

Since σ_ξ, σ_ζ, $\tau_{\xi\zeta}$ and σ_θ in Eq. (24) satisfying the equilibrium equation must also satisfy the compatibility equations, c_1 and c_2 must be found from the compatibility equations. Thus, the use of the first compatibility equation gives

$$c_1=-\frac{1}{2}\left[\frac{[(2+\nu)\mu_1^2+(1+\nu)\mu_2^2+\mu_4^2]A_o}{16}+\left[(2+\nu)+(1+\nu)\frac{a^2}{b^2}\right]\right]. \quad (26)$$

The second compatibility equation gives

$$c_2=-\frac{a^2}{2b^2}-\frac{A_o}{32}[(2+\nu)\mu_1^2+(1+\nu)\mu_4^2+\mu_2^2]. \quad (27)$$

Substituting these values of c_1 and c_2 in the expression for c_0, we get

$$c_o=\frac{1}{8(ab-cd)}\Bigg\{(ba^3-dc^3)\times\Bigg\{\frac{[(2+\nu)\mu_1^2+(1+\mu)\mu_2^2+\mu_4^2]A_o}{16}$$

$$+\left[(2+\nu)+(1+\nu)\frac{a^2}{b^2}\right]\Bigg\}$$
$$+(ab^3-cd^3)\left\{\frac{a^2}{b^2}+\frac{A_o}{16}[(2+\nu)\mu_1^2\right.$$
$$\left.+(1+\nu)\mu_4^2+\mu_2^2]\right\}\Bigg\}. \quad (28)$$

c will be obtained from the moment equation

$$M=\iint -\sigma_\theta \xi d\xi d\zeta \quad (29)$$

or

$$M_o=\iint -\sigma_{\theta_0}\xi d\xi d\zeta=\iint cE\xi^2 d\xi d\zeta$$
$$=cE\pi\left(\frac{ba^3}{4}-\frac{dc^3}{4}\right), \quad (30)$$

from which

$$cE=\frac{4M_o}{\pi(ba^3-dc^3)}. \quad (31)$$

Thus, σ_θ can be found from

$$\sigma_\theta=\sigma_{\theta_0}+\sigma_{\theta_1}=-cE\xi+\frac{cE}{R}(c_o+c_1\xi^2+c_2\zeta^2). \quad (32)$$

3. Results and Discussion

3.1 Circular toroidal ring sector

The formulas given above can be transformed into the case of the circular toroidal ring sector with the section composed of two circles by inserting $a=b$ and $c=d$ in the above equations. In particular, with the purpose of comparing the results given here with those obtained by Lang[(1)] and Göhner[(4)] for the circular cross sections, we will take Poisson's ratio as $\nu=0.3$, and compare the stresses at the inner point $\xi=a$. In this case, the constants are found to be $c_0=0.22115(a^2+c^2)$, $A_0=-1.23076/(a^2c^2)$, $c_1=-1.09231$, and $c_2=0.20769$. Then, Eq. (32) gives

$$\sigma_\theta=\frac{-4Ma}{\pi(a^4-c^4)}\left[1+0.87115\frac{a}{R}-0.22115\frac{c^2}{Ra}\right] \quad (33)$$

at the point ($\xi=a$, $\zeta=0$). This formula in a generalization of the formula given for σ_θ in Ref. (1) for the elliptical ring sectors with the section composed of two confocal ellipses. In this equation, by putting $c=0$, we can readily obtain the same results predicted by Göhner[(4)] and Lang[(1)]. Since Eq. (33) can be used not only for toroidal rings with solid cross sections but also for toroidal rings with cross sections composed of two confocal ellipses or two concentric circles ($a=b$, $c=d$), it enables us to analyze elbow pipe elements and to compare the theoretical results with those predicted by numerical techniques[(5)(6)(9)]. It is seen from Eq. (33) that σ_θ decreases with the enlargement of the circular gap. In the same way, it is clear that σ_θ decreases with an increase in R. In the case of a straight bar with a solid cross section ($R\to\infty$), Eq. (33) gives the same result in the elementary strength of materials, $\sigma_\theta=-4M/\pi a^3$, which is the stress distribution for pure bending of prismatic bars. From this result, we observe that the formula given by Lang[(1)] is wrong. Indeed, it can easily be seen from Lang's results[(1)] that the equations given for stresses do not have the dimension of stress (N/m^2, N: Newton, m: meter).

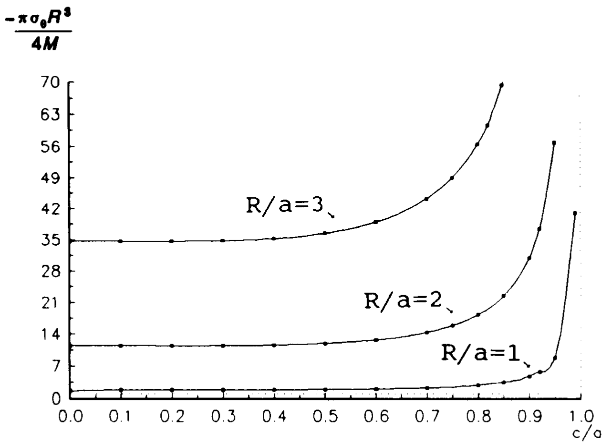


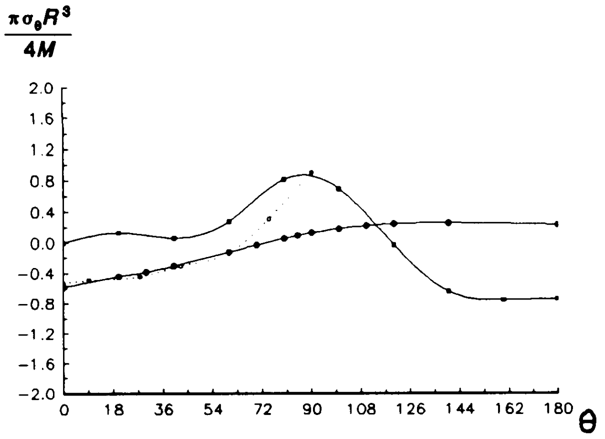
Fig. 2 Theoretical results for circumferential stress versus c/a

We can rewrite Eq. (33) depending upon the ratios c/a and R/a as

$$\sigma_\theta=\frac{-4M}{\pi a^2\left[1-\left(\frac{c}{a}\right)^4\right]R}\left[\frac{R}{a}+0.87115-0.22115\left(\frac{c}{a}\right)^2\right]. \quad (34)$$

Figure 2 shows the plots of $-\pi\sigma_\theta R^3/4M$ versus c/a for various values of R/a. We see from this figure that the circumferential stress σ_θ shows small changes with increasing values of c/a. For $c/a\to 1.0$, $-\pi\sigma_\theta R^3/4M$ goes to infinity, which means that the thickness goes to zero. The values on the y axis ($c/a=0$) correspond to the case of solid cross sections. On the other hand, we observe that $-\pi\sigma_\theta R^3/4M$ varies sharply for different values of R/a, although the inclination of each curve corresponding to different values of R/a does not change appreciably. Thus, we can say that the most important physical quantity of the dimensions which effects the stress value is not the ratio of the lengths of principal axes of the ellipses, but the quantity R/a, the ratio of curvature to the length of the principal axis of the outer ellipse of the section.

Figure 3 shows the variation of σ_θ with the angle θ measured as in Fig. 1(b) in the interval 0°−180°. As is seen, the theoretical value of σ_θ is maximum at $\theta=180°$, as expected. The points • and ∘ correspond to the theoretical and experimental values given in Ref. (5). It is seen from this figure that the theoretical and experimental results are in good agreement in the interval 0°−90°. Although the experimental and the theoretical results are given for the value of $\theta=45°$ in Ref. (5), this angle does not form any change on the shape of the present curve since it is assumed in this study that the stress distribution is not affected



$R=250$ mm, $\delta=12.5$ mm,
$E=200$ Gpa, $\nu=0.28$, $2a=275$ mm
· Theoretical Stresses
◦ Experimental Stresses obtained by Whatham[9]
■ Stresses by Bathe and Almeida[5]

Fig. 3 Comparison of the theoretical, experimental and numerical results for the circumferential stress distribution at the point $\xi=a$ in a circular ring sector ($c=d=0$). The modulus of elasticity and the Poisson's ratio are taken as 200 Gpa and 0.28, respectively.

by the angle θ because of the symmetry in the ring plane. Thus, the present curve shows the stresses on all points where $\xi=a$.

4. Conclusions

In this study, the analytical expressions are derived for the stresses in a toroidal ring sector with a cross section of two confocal ellipses. In the special case, these results have been reduced to the problem of pure bending of a ring sector of circular cross section. A comparison of the results obtained with the experimental and numerical results has shown their good agreement. The analysis has been limited to determining the first correction to the initial stress state for pure bending of a ring sector with the cross section of two confocal ellipses, since the second correction would necessitate the determination of complex and long equations. On the other hand, by reducing the equations to the case of solid circular cross sections ($c=d=0$, $a=b$), the same results as those given for σ_θ by Göhner[4] have been obtained. It has also been shown that the theoretical results given for σ_θ by Lang[1] were dimensionally incorrect. We keep in mind here that the method developed is also suitable for analyzing the toroidal ring sectors whose cross sections are surrounded by an outer ellipse and an inner circular gap along the ring. In this case, it will be sufficient to use $c=d$ in the equations.

It must be noted here that a more complete theory can be constructed with the inclusion of the assumption that ξ/R is not so small that it can be neglected in the theory. However, such an analysis will probably necessitate the use of numerical techniques.

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