PRODUCT SUBMANIFOLDS WITH POINTWISE 3-PLANAR NORMAL SECTIONS

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0. Introduction. Let M be a smooth m-dimensional submanifold in $(m+d)$-dimensional Euclidean space $\mathbb{R}^{m+d}$. For $x \in M$ and a non-zero vector X in T_xM, we define the $(d+1)$-dimensional affine subspace $E(x, X)$ of $\mathbb{R}^{m+d}$ by

$$E(x, X) = x + \text{span}\{X, N_x(M)\}.$$

In a neighbourhood of x, the intersection $M \cap E(x, X)$ is a regular curve $\gamma:(-\varepsilon, \varepsilon) \to M$. We suppose the parameter $t \in (-\varepsilon, \varepsilon)$ is a multiple of the arc-length such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Each choice of $X \in T(M)$ yields a different curve which is called the *normal section* of M at x in the direction of X, where $X \in T_x(M)$ (Section 3).

For such a normal section we can write

$$\gamma(t) = x + \lambda(t)X + N(t). \tag{0.1}$$

where $N(t) \in N_x(M)$ and $\lambda(t) \in \mathbb{R}$.

The submanifold M is said to have *pointwise k-planar normal sections* (*Pk-PNS*) if for each normal section γ the first, second and higher order derivatives

$$\{\dot{\gamma}(0), \ddot{\gamma}(0), \dddot{\gamma}(0), \ldots, \gamma^{(k)}(0)\}$$

are linearly dependent as vectors in $\mathbb{R}^{n+d}$.

Submanifolds with pointwise 3-planar normal sections have been studied by S. J. Li in the case when M is isotropic [**6**] and also in the case when M is spherical [**7**].

In this paper we consider product submanifolds $M = M_1 \times M_2$ with P3-PNS and we show that this implies strong conditions on M_1 and M_2.

1. Basic notation. Let M be an m-dimensional submanifold in $(m+d)$-dimensional Euclidean space $\mathbb{R}^{m+d}$. Let ∇ and D denote the covariant derivatives in M and $\mathbb{R}^{m+d}$ respectively. Thus D_X is just the directional derivative in the direction X in $\mathbb{R}^{m+d}$. Then for tangent vector fields X, Y and Z over M we have

$$D_XY = \nabla_XY + h(X, Y)$$

where h is the *second fundamental form* of M. We define $\bar{\nabla}_Xh$ as usual by

$$\bar{\nabla}_X(h(Y, Z)) = (\bar{\nabla}_Xh)(Y, Z) + h(\nabla_XY, Z) + h(Y, \nabla_XZ), \tag{1.1}$$

Then we have

$$h(X, Y) = h(Y, X) \tag{1.2}$$

and

$$\begin{aligned}(\bar{\nabla}_Xh)(Y, Z) &= (\bar{\nabla}_Yh)(X, Z) = (\bar{\nabla}_Zh)(X, Y) = (\bar{\nabla}_Xh)(Z, Y)\\ &= (\bar{\nabla}_Yh)(Z, X) = (\bar{\nabla}_Zh)(Y, X).\end{aligned} \tag{1.3}$$

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We will also need to refer to the *shape operator* A_v where this is defined for a normal field $v: M \to N(M)$ by $\langle h(X, Y), v\rangle = \langle A_v X, Y\rangle$. Note however that $h(X, Y)$, $(\nabla_X h)(Y, Z)$ and $A_v X$ have values at $x \in M$ which depend only on the values of X, Y, Z and v at $x \in M$ and not on their derivatives.

2. Preliminary results. Let us write

$$H(X) = h(X, X),$$

$$\nabla H(X) = (\bar{\nabla}_X h)(X, X)$$

$$J(X) = (\bar{\nabla}_X \bar{\nabla}_X h)(X, X) + 3h(A_{h(X,X)}X, X)$$

so that $H: T(M) \to N(M)$ and $\nabla H: T(M) \to N(M)$ and $J: T(M) \to N(M)$ are fibre maps whose restriction to each fibre $T_x(M)$ is a homogeneous polynomial map, H is of degree 2 and ∇H is of degree 3 and J is of degree 4.

Note that (1.2) shows that h is completely determined by H and (1.3) shows that $\bar{\nabla} h$ is completely determined by ∇H.

LEMMA 2.1. *M has* P3-PNS *if and only if for all $x \in M$ and $X \in T_x(M)$, $H(X)$, $\nabla H(X)$ and $J(X)$ are linearly dependent vectors in $N_x(M)$.*

Proof. Differentiating the formula (0.2) and evaluating at $t = 0$ we obtain after some calculation

$$\dot{\gamma}(0) = X, \dot{N}(0) = 0$$

$$\ddot{\gamma}(0) = h(X, X) = \ddot{N}(0)$$

$$\dddot{\gamma}(0) = \nabla_X \nabla_X X - A_{h(X,X)}X + (\bar{\nabla}_X h)(X, X) = \dddot{\lambda}(0)X + \dddot{N}(0)$$

$$\gamma^{(iv)}(0) = (\textit{terms in } T_x(M)) + 4\lambda^{(iv)}(0)h(X, X) +$$

$$+ 3h(A_{h(X,X)}X, X) + (\bar{\nabla}_X \bar{\nabla}_X h)(X, X) =$$

$$= \lambda^{(iv)}(0)X + N^{(iv)}(0).$$

From this it is clear that $\{\dot{\gamma}(0), \ddot{\gamma}(0), \dddot{\gamma}(0), \gamma^{(iv)}(0)\}$ is a linearly dependent set if and only if $\{\ddot{N}(0), \dddot{N}(0), N^{(iv)}(0)\}$ is a linearly dependent set. It is also clear that $\ddot{N}(0) = H(X)$, $\dddot{N}(0) = \nabla H(X)$, $N^{(iv)}(0) = 4\lambda^{(iv)}(0)H(X) + J(X)$. Hence $\{\ddot{N}(0), \dddot{N}(0), N^{(iv)}(0)\}$ is a linearly dependent set if and only if $\{H(X), \nabla H(X), J(X)\}$ is a linearly dependent set.

LEMMA 2.2. *M has* P2-PNS *if and only if*

$$\|H\|^2 \nabla H - \langle H, \nabla H\rangle H \equiv 0. \tag{2.1}$$

Proof. $\langle H, \nabla H\rangle \nabla H - \langle H, \nabla H\rangle H \equiv 0$ means that for all $x \in M$ and $X \in T_x(M)$,

$$\|H(X)\|^2 \nabla H(X) - \langle H(X), \nabla H(X)\rangle H(X) = 0. \tag{2.2}$$

Suppose M has P2-PNS, then for all $X \in T(M)$, $H(X)$ and $\nabla H(X)$ are linearly dependent. So either $\|H(X)\| = 0$ or for some $\alpha \in \mathbb{R}$, $\nabla H(x) = \alpha H(X)$. In either case it is easy to check that (2.2) holds. The converse is obvious.

THEOREM 2.3. *M has* P3-PNS *if and only if*

$$\{\|H\|^2\,\|\nabla H\|^2 - \langle H, \nabla H\rangle^2\}J$$
$$\equiv \{\langle J, H\rangle \|\nabla H\|^2 - \langle J, \nabla H\rangle\langle H, \nabla H\rangle\}H + \{\langle J, \nabla H\rangle\,\|H\|^2 - \langle J, H\rangle\langle \nabla H, H\rangle\}\nabla H. \tag{2.3}$$

Proof. Observe that

$$\|\,\|H\|^2\,\nabla H - \langle H, \nabla H\rangle H\|^2 = \|H\|^2\{\|H\|^2\,\|\nabla H\|^2 - \langle H, \nabla H\rangle^2\}.$$

Again the condition (2.3) means that for all $X \in T(M)$

$$\{\|H(X)\|^2\,\nabla H(X) - \langle H(X), \nabla H(X)\rangle^2\}J(X)$$
$$= \{\langle J(X), H(X)\rangle\,\|\nabla H(X)\|^2 - \langle J(X), \nabla H(X)\rangle\langle H(X), \nabla H(X)\rangle\}H(X)$$
$$+ \{\langle J(X), \nabla H(X)\rangle\,\|H(X)\|^2 - \langle J(X), H(X)\rangle\langle \nabla H(X), H(X)\rangle\}\nabla H(X). \tag{2.4}$$

In this case the condition that $H(X)$, $\nabla H(X)$, $J(X)$ are linearly dependent can be written as either

(a) $H(X)$, $\nabla H(X)$ are linearly dependent or

(b) $J(X) = \alpha H(X) + \beta \nabla H(X)$ for some α, $\beta \in \mathbb{R}$.

So in case (a) we can use Lemma (2.2) to see that (2.4) is true since the coefficients of $H(X)$, $\nabla H(X)$ and $J(X)$ are all zero. In case (b) one checks that (2.4) is true by substituting for $J(X)$.

Conversely if (2.4) is true, this says that $H(X)$, $\nabla H(X)$, $J(X)$ are linearly dependent unless the coefficients are all zero. However in the latter case, again by Lemma (2.2), $H(X)$ and $\nabla H(X)$ are linearly dependent; which of course means that $H(X)$, $\nabla H(X)$, $J(X)$ are linearly dependent.

Submanifolds with P2-PNS have been classified. The subclass of those submanifolds with P2-PNS for which $\nabla H \equiv 0$, that is, that have parallel second fundamental form have been shown to coincide with the class of extrinsically symmetric submanifolds which were classified by Ferus. We will call these s-submanifolds. We have shown in another paper that the other manifolds with P2-PNS must be hypersurfaces.

In the classification theorem we give below we separate manifolds into three types.

DEFINITION 2.4. Submanifolds are

(i) *of type* AW(1) if they satisfy $J \equiv 0$;

(ii) *of type* AW(2) if they satisfy $\|\nabla H\|^2 J \equiv \langle J, \nabla H\rangle \nabla H$;

(iii) *of type* AW(3) if they satisfy $\|H\|^2 J = \langle J, H\rangle H$.

We have not investigated yet the geometrical consequences of these conditions. Note however that (ii) is the condition for ∇H and J to be linearly dependent and (iii) is the condition for H and J to be linearly dependent, so if any one of these three conditions hold M will have P3-PNS.

Note also that for a submanifold with P2-PNS either $\nabla H \equiv 0$, when it is an s-submanifold and it is automatically of type AW(2) or by the theorem in **[1]** it is a hypersurface and so automatically of type AW(2) and of type AW(3).

3. Product submanifolds with P3-PNS. Now consider the case when $M = M_1 \times M_2$ is a product submanifold. That is, there exist isometric embeddings $f_1 : M_1 \to \mathbb{R}^{m_1+d_1}$ and $f_2 : M_2 \to \mathbb{R}^{m_2+d_2}$; we put $m = m_1 + m_2$, $d = d_1 + d_2$ so that $\mathbb{R}^{m+k} \equiv \mathbb{R}^{m_1+d_1} \times \mathbb{R}^{m_2+d_2}$ and then

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2))$$

defines the embedding $f : M \to \mathbb{R}^{m+d}$ which we will take to be the inclusion map. With this abuse of notation we can write, for any $x \in M$, $x = (x_1, x_2)$

$$\left.\begin{aligned} T_x(M) &= T_{x_1}(M) \oplus T_{x_2}(M) \\ N_x(M) &= N_{x_1}(M) \oplus N_{x_2}(M) \end{aligned}\right\} \tag{3.1}$$

We cite here, in a slightly modified form, a theorem due to Deprez and Verheyen for submanifolds with P2-PNS. The main purpose of this paper is to generalise this theorem.

THEOREM. *Let $M = M_1 \times M_2$ be a product submanifold of $\mathbb{R}^{m+d}$ where $M_1 \subset \mathbb{R}^{m_1+d_1}$, $M_2 \subset \mathbb{R}^{m_2+d_2}$. Then $M_1 \times M_2$ has* P2-PNS *if and only if M_1 and M_2 are s-submanifolds or one of them is totally geodesic and the other is either an s-submanifold or a hypersurface.*

Proof. In [**4**] it is proved that $M_1 \times M_2$ has P2-PNS if and only if either both M_1 and M_2 have parallel second fundamental form, or one of them is totally geodesic and the other has P2-PNS. However, Ferus [**5**] has shown that submanifolds with parallel second fundamental form are s-submanifolds and we have shown that if a manifold has P2-PNS, it either has parallel second fundamental form (and hence is an s-submanifold) or it is a hypersurface [**1**].

CLASSIFICATION THEOREM. *Let $M = M_1 \times M_2$ be a product submanifold of $\mathbb{R}^{m+d}$ where $M_1 \subset \mathbb{R}^{m_1+d_1}$, $M_2 \subset \mathbb{R}^{m_2+d_2}$. Then $M_1 \times M_2$ has* P3-PNS *if and only if, up to an interchange of M_1 and M_2, one of the following is true:*

I. *M_1 is totally geodesic and M_2 has* P3-PNS;

II. (a) *M_1 is an s-submanifold of type* AW(1) *and M_2 is a submanifold of type* AW(2);

(b) *M_1 is a submanifold of type* AW(1) *with* P2-PNS *and M_2 is a submanifold of type* AW(1);

III. *M_1 is an s-submanifold of type* AW(3) *and M_2 has* P2-PNS;

IV. *M_1 and M_2 are both submanifolds of type* AW(3) *with* P2-PNS.

Proof. Let h_1 and h_2 be the second fundamental forms of M_1 and M_2. We define H_1, H_2, ∇H_1, ∇H_2, J_1, J_2 analogously for M_1 and M_2 in terms of h_1, h_2 and their covariant derivatives. Then if $X \in T_x(M)$, where $x = (x_1, x_2)$, we can write $X = X_1 \oplus X_2$, where $X_1 \in T_{x_1}(M_1)$, $X_2 \in T_{x_2}(M_2)$ and it is easy to see that

$$H(X) = H_1(X_1) \oplus H_2(X_2),$$

$$\nabla H(X) = \nabla H_1(X_1) \oplus \nabla H_2(X_2)$$

and

$$J(X) = J_1(X_1) \oplus J_2(X_2).$$

We will express this by saying that

$$H = H_1 + H_2,$$

$$\nabla H = \nabla H_1 + \nabla H_2$$

and

$$J = J_1 + J_2.$$

For convenience let us use a shorter notation for the coefficients in Theorem 2.3 and put

$$\alpha = \langle J, \{\|\nabla H\|^2 H - \langle H, \nabla H\rangle \nabla H\}\rangle,$$
$$\beta = \langle J, \{\|H\|^2 \nabla H - \langle H, \nabla H\rangle H\}\rangle,$$
$$\delta = \|H\|^2 \|\nabla H\|^2 - \langle H, \nabla H\rangle^2.$$

We define α_1, β_1, δ_1 and α_2, β_2, δ_2 similarly for M_1 and M_2. Then we can write

$$\begin{aligned}
\alpha &= \langle J_1 + J_2, H_1 + H_2\rangle \|\nabla H_1 + \nabla H_2\|^2 - \langle J_1 + J_2, \nabla H_1 + \nabla H_2\rangle\langle H_1 + H_2, \nabla H_1 + \nabla H_2\rangle \\
&= \langle J_1, H_1\rangle \|\nabla H_1\|^2 - \langle J_1, \nabla H_1\rangle\langle H_1, \nabla H_1\rangle \\
&\quad + \langle J_2, H_2\rangle \|\nabla H_2\|^2 - \langle J_2, \nabla H_2\rangle\langle H_2, \nabla H_2\rangle \\
&\quad + \langle J_1, H_1\rangle \|\nabla H_2\|^2 - \langle J_1, \nabla H_1\rangle\langle H_2, \nabla H_2\rangle \\
&\quad + \langle J_2, H_2\rangle \|\nabla H_1\|^2 - \langle J_2, \nabla H_2\rangle\langle H_1, \nabla H_1\rangle \\
&= \alpha_1 + \alpha_2 + \langle J_1, H_1\rangle \|\nabla H_2\|^2 - \langle J_1, \nabla H_1\rangle\langle H_2, \nabla H_2\rangle \\
&\quad + \langle J_2, H_2\rangle \|\nabla H_1\|^2 - \langle J_2, \nabla H_2\rangle\langle H_1, \nabla H_1\rangle
\end{aligned}$$

and similarly

$$\begin{aligned}
\beta &= \langle J_1 + J_2, \nabla H_1 + \nabla H_2\rangle \|H_1 + H_2\|^2 - \langle J_1 + J_2, H_1 + H_2\rangle\langle \nabla H_1 + \nabla H_2, H_1 + H_2\rangle \\
&= \langle J_1, \nabla H_1\rangle \|H_1\|^2 - \langle J_1, H_1\rangle\langle \nabla H_1, H_1\rangle \\
&\quad + \langle J_2, \nabla H_2\rangle \|H_2\|^2 - \langle J_2, H_2\rangle\langle \nabla H_2, H_2\rangle \\
&\quad + \langle J_2, \nabla H_2\rangle \|H_1\|^2 - \langle J_2, H_2\rangle\langle \nabla H_1, H_1\rangle \\
&\quad + \langle J_1, \nabla H_1\rangle \|H_2\|^2 - \langle J_1, H_1\rangle\langle \nabla H_2, H_2\rangle \\
&= \beta_1 + \beta_2 + \langle J_2, \nabla H_2\rangle \|H_1\|^2 - \langle J_2, H_2\rangle\langle \nabla H_1, H_1\rangle \\
&\quad + \langle J_1, \nabla H_1\rangle \|H_2\|^2 - \langle J_1, H_1\rangle\langle \nabla H_2, H_2\rangle
\end{aligned}$$

and similarly

$$\begin{aligned}
\delta &= \|H_1 + H_2\|^2 \|\nabla H_1 + \nabla H_2\|^2 - \langle H_1 + H_2, \nabla H_1 + \nabla H_2\rangle\langle H_1 + H_2, \nabla H_1 + \nabla H_2\rangle \\
&= \|H_1\|^2 \|\nabla H_1\|^2 - \langle H_1, \nabla H_1\rangle\langle H_1, \nabla H_1\rangle \\
&\quad + \|H_2\|^2 \|\nabla H_2\|^2 - \langle H_2, \nabla H_2\rangle\langle H_2, \nabla H_2\rangle \\
&\quad + \|H_1\|^2 \|\nabla H_2\|^2 - \langle H_1, \nabla H_1\rangle\langle H_2, \nabla H_2\rangle \\
&\quad + \|H_2\|^2 \|\nabla H_1\|^2 - \langle H_2, \nabla H_2\rangle\langle H_1, \nabla H_1\rangle \\
&= \delta_1 + \delta_2 + \|H_1\|^2 \|\nabla H_2\|^2 - 2\langle H_1, \nabla H_1\rangle\langle H_2, \nabla H_2\rangle + \|H_2\|^2 \|\nabla H_1\|^2.
\end{aligned}$$

Now M has P3-PNS if and only if for all $X_1 \in T(M_1)$, $X_2 \in T(M_2)$

$$\begin{aligned}
\alpha(X_1 + X_2)H(X_1 + X_2) &+ \beta(X_1 + X_2)\nabla H(X_1 + X_2) \\
&= \delta(X_1 + X_2)J(X_1 + X_2).
\end{aligned}$$

From the above equations we have

$$\begin{aligned}
\alpha H + \beta\nabla H + \delta J \equiv\ & \alpha_1 H_1 + \beta_1\nabla H_1 - \delta_1 J_1 \\
& + \alpha_2 H_2 + \beta_2\nabla H_2 - \delta_2 J_2 \\
& + \alpha_1 H_2 + \beta_1\nabla H_2 - \delta_1 J_2 \\
& + \alpha_2 H_1 + \beta_2\nabla H_1 - \delta_2 J_1 \\
& + \langle J_1, \nabla H_1\rangle\{\|H_2\|^2\nabla H_2 - \langle H_2, \nabla H_2\rangle H_2\} \\
& + \langle J_2, \nabla H_2\rangle\{\|H_1\|^2\nabla H_1 - \langle H_1, \nabla H_1\rangle H_1\} \\
& + \langle J_2, H_2\rangle\{\|\nabla H_1\|^2 H_1 - \langle H_1, \nabla H_1\rangle\nabla H_1\} \\
& - \langle J_1, H_1\rangle\{\|\nabla H_2\|^2 H_2 - \langle H_2, \nabla H_2\rangle\nabla H_2\} \\
& - \|\nabla H_1\|^2\{\|H_2\|^2 J_2 - \langle J_2, H_2\rangle H_2\} \\
& - \|\nabla H_2\|^2\{\|H_1\|^2 J_1 - \langle J_1, H_1\rangle H_1\} \\
& + \langle H_1, \nabla H_1\rangle\{2\langle H_2, \nabla H_2\rangle J_2 - \langle J_2, H_2\rangle\nabla H_2 - \langle J_2, \nabla H_2\rangle H_2\} \\
& + \langle H_2, \nabla H_2\rangle\{2\langle H_1, \nabla H_1\rangle J_1 - \langle J_1, H_1\rangle\nabla H_1 - \langle J_1, \nabla H_1\rangle H_1\} \\
& - \|H_1\|^2\{\|\nabla H_2\|^2 J_2 - \langle J_2, \nabla H_2\rangle\nabla H_2\} \\
& - \|H_2\|^2\{\|\nabla H_1\|^2 J_1 - \langle J_1, \nabla H_1\rangle\nabla H_1\} = 0.
\end{aligned}$$

If we fix $x \in M$, then α_1, β_1, δ_1 are polynomials on $T_{x_1}(M_1)$ and α_2, β_2, δ_2 are polynomials on $T_{x_2}(M_2)$ where $x = (x_1, x_2)$. Similarly $H_1, \nabla H_1, J_1$ are polynomial maps from $T_{x_1}(M_1)$ to $N_{x_1}(M_1)$ and H_2, ∇H_2, J_2 are polynomial maps from $T_{x_2}(M_2)$ to $N_{x_2}(M_2)$. So if we look at the above equation and think about the degrees of the terms as polynomials in X_1 (or X_2), and then pick out terms of the same degree in X_i, $i = 1, 2$, we get the following:

(a) $\alpha_1 H_1 + \beta_1\nabla H_1 \equiv \delta_1 J_1, \quad \alpha_2 H_2 + \beta_2\nabla H_2 \equiv \delta_2 J_2,$
(b) $\alpha_1 H_2 \equiv 0, \quad \alpha_2 H_1 \equiv 0,$
(c) $\beta_1\nabla H_2 \equiv 0, \quad \beta_2\nabla H_1 \equiv 0,$
(d) $\delta_1 J_2 \equiv 0, \quad \delta_2 J_1 \equiv 0,$
(e) $\langle J_1, \nabla H_1\rangle\{\|H_2\|^2\nabla H_2 - \langle H_2, \nabla H_2\rangle H_2\} \equiv 0,$
$\langle J_2, \nabla H_2\rangle\{\|H_1\|^2\nabla H_1 - \langle H_1, \nabla H_1\rangle H_1\} \equiv 0,$
(f) $\langle J_1, H_1\rangle\{\|\nabla H_2\|^2 H_2 - \langle H_2, \nabla H_2\rangle\nabla H_2\} - \|\nabla H_1\|^2\{\|H_2\|^2 J_2 - \langle H_2, J_2\rangle H_2\} \equiv 0,$
$\langle J_2, H_2\rangle\{\|\nabla H_1\|^2 H_1 - \langle H_1, \nabla H_1\rangle\nabla H_1\} - \|\nabla H_2\|^2\{\|H_1\|^2 J_1 - \langle H_1, J_1\rangle H_1\} \equiv 0,$
(g) $\langle H_1, \nabla H_1\rangle\{2\langle H_2, \nabla H_2\rangle J_2 - \langle J_2, H_2\rangle\nabla H_2 - \langle J_2, \nabla H_2\rangle H_2\} \equiv 0,$
$\langle H_2, \nabla H_2\rangle\{2\langle H_1, \nabla H_1\rangle J_1 - \langle J_1, H_1\rangle\nabla H_1 - \langle J_1, \nabla H_1\rangle H_1\} \equiv 0,$
(h) $\|H_1\|^2\{\|\nabla H_2\|^2 J_2 - \langle J_2, \nabla H_2\rangle\nabla H_2\} \equiv 0,$
$\|H_2\|^2\{\|\nabla H_1\|^2 J_1 - \langle J_1, \nabla H_1\rangle\nabla H_1\} \equiv 0.$

We have also borne in mind that H_1, ∇H_1, J_1 have values in $N(M_1)$ and H_2, ∇H_2, J_2 have values in $N(M_2)$ so that the terms of degree 10 actually give (d) and (h).

Now (a) means that both M_1 and M_2 have P3-PNS. Then from (b) we see that at each point $(x_1, x_2) \in M_1 \times M_2 = M$ we have either $\alpha_1 = 0$ or $H_2 = 0$. But α_1 depends only on x_1 and not on x_2. So if $\alpha_1 \neq 0$ at x_1 we must have $H_2 \equiv 0$ for all $x_2 \in M_2$. Thus either $\alpha_1 \equiv 0$ or $H_2 \equiv 0$. A similar argument applies to (c) and (d). Thus we distinguish the following cases:

Case I: $H_1 \equiv 0$ or $H_2 \equiv 0$;

Case II: $\alpha_1 \equiv 0$ and $\alpha_2 \equiv 0$ and either $J_1 \equiv 0$ or $J_2 \equiv 0$;

Case III: α_1, α_2, δ_1, δ_2 are all identically zero and either $\nabla H_1 \equiv 0$ or $\nabla H_2 \equiv 0$;

Case IV: α_1, α_2, β_1, β_2, δ_1, δ_2 are all identically zero. Note that $\delta_i \equiv 0$ implies that $\alpha_i \equiv 0$ and $\beta_i \equiv 0$.

Let us consider these in turn.

Case I. Suppose $H_1 \equiv 0$. This implies that the second fundamental form on M_1 is identically zero. Thus M_1 is totally geodesic and α_1, β_1, δ_1, H_1, ∇H_1, J_1 are all identically zero. Thus conditions (b) to (h) are automatically satisfied. Thus in this case M_1 is totally geodesic and M_2 has P3–PNS and this is sufficient to ensure that $M_1 \times M_2$ has P3-PNS.

Case II. Suppose that $J_1 \equiv 0$. This implies that $\beta_1 \equiv 0$. Note also that $\nabla H_1 \equiv 0$ implies $\delta_1 \equiv 0$. We use (c) to distinguish the subcases.

Case II(a): α_1, α_2, J_1, ∇H_1 are identically zero.

Case II(b): α_1, α_2, β_1, β_2, J_1, are identically zero.

Case II(c): α_1, α_2, J_1, ∇H_2 are identically zero.

Case II(a). In this case the only condition which remains to be satisfied is

$$\|H_1\|^2\{\|\nabla H_2\|^2 J_2 - \langle J_2, \nabla H_2\rangle \nabla H_2\} \equiv 0,$$

since $\nabla H_1 \equiv 0$ implies $\beta_1 \equiv 0$ and $\delta_1 \equiv 0$. Thus we either have Case I or

$$\|\nabla H_2\|^2 J_2 - \langle J_1, \nabla H_2\rangle \nabla H_2 \equiv 0,$$

which implies that $\alpha_2 \equiv 0$. M_1 is an s-submanifold of type AW(1) and M_2 is a type AW(2) submanifold. This is sufficient to ensure that $M_1 \times M_2$ has P3-PNS.

Case II(b). Note first of all that for any submanifold M, $\alpha = \beta = 0$ implies that either $\delta = 0$ or $\langle J, H\rangle = \langle J, \nabla H\rangle = 0$. This is because if $\delta \neq 0$, then ∇H and H are linearly independent and $\|H\|^2\|\nabla H\|^2 - \langle H, \nabla H\rangle > 0$. Hence $\|H\|^2 \nabla H - \langle H, \nabla H\rangle H$ and $\|\nabla H\|^2 H - \langle H, \nabla H\rangle \nabla H$ are also linearly independent. But $\alpha = \beta = 0$ says that J is perpendicular to both of these vectors and hence to the plane spanned by H and ∇H. Hence $\langle J, H\rangle = \langle J, \nabla H\rangle = 0$.

Now in our case $\delta_1 J_2 \equiv 0$, so either $\delta_1 \equiv 0$ or $J_2 \equiv 0$. But if $\delta_1 \equiv 0$, then one of the remaining conditions to be satisfied is

$$\|\nabla H_1\|^2\{\|H_2\|^2 J_2 - \langle H_2, J_2\rangle H_2\} \equiv 0.$$

Here $\nabla H_1 \equiv 0$ gives us the Case II(a) and because $\alpha_2 = \beta_2 = 0$, then either $\delta_2 = 0$, which gives Case IV, or by the above

$$\langle J_2, H_2\rangle = \langle J_2, \nabla H_2\rangle = 0.$$

So $\|H\|^2 J_2 = 0$. But $H_2 = 0$ gives the Case I and $J_2 = 0$ says M_2 has the property AW(1). So in this case M_1 is a submanifold with P2-PNS satisfying the condition AW(1) and M_2 is a submanifold with P3-PNS satisfying the condition AW(1). This is sufficient to ensure that $M_1 \times M_2$ has P3-PNS.

Case II(c). Note that $\nabla H_2 \equiv 0$ implies that $\beta_2 \equiv 0$ and $\delta_2 \equiv 0$ and $J_1 \equiv 0$ implies

$\beta_1 \equiv 0$. We must have $\delta_1 J_2 \equiv 0$. So either $\delta_1 \equiv 0$ or $J_2 \equiv 0$. But if $\delta_1 \equiv 0$, we have Case IV, which we consider later. If we suppose $J_2 \equiv 0$, then we have Case II(a) (but with M_1 and M_2 interchanged). So this gives nothing new.

Case III. Suppose that $\nabla H_1 \equiv 0$ and α_1, α_2, δ_1, δ_2 are all identically zero. Then $\nabla H_1 \equiv 0$ implies that $\alpha_1 \equiv 0$ and $\beta_1 \equiv 0$. The only conditions that have to be satisfied are

$$\|\nabla H_2\|^2 \{\|H_1\|^2 J_1 - \langle H_1, J_1\rangle H_1\} \equiv 0,$$

$$\|H_1\|^2 \{\|\nabla H_2\|^2 J_2 - \langle J_2, H_2\rangle H_2\} \equiv 0.$$

Thus we either have Case I or $\|\nabla H_2\|^2 J_2 - \langle J_2, \nabla H_2\rangle \nabla H_2\} \equiv 0$ which implies $\alpha_2 \equiv 0$ and $\beta_2 \equiv 0$. Thus in this case M_1 is an s-submanifold of type AW(3) and M_2 is a submanifold of type AW(2) with P2-PNS. This is sufficient for $M_1 \times M_2$ to have P3-PNS.

Case IV. The conditions that remain to be satisfied are

(f) $\|\nabla \mathrm{H}_1\|^2 \{\|H_2\|^2 J_2 - \langle H_2, J_2\rangle H_2\} \equiv 0,$
$\|\nabla H_2\|^2 \{\|H_1\|^2 J_1 - \langle H_1, J_1\rangle H_1\} \equiv 0,$
(g) $\langle H_1, \nabla H_1\rangle\{2\langle H_2, \nabla H_2\rangle J_2 - \langle J_2, H_2\rangle \nabla H_2 - \langle J_2, \nabla H_2\rangle H_2\} \equiv 0,$
$\langle H_2, \nabla H_2\rangle\{2\langle H_1, \nabla H_1\rangle J_1 - \langle J_1, H_1\rangle \nabla H_1 - \langle J_1, \nabla H_1\rangle H_1\} \equiv 0,$
(h) $\|H_1\|^2 \{\|\nabla H_2\|^2 J_2 - \langle J_2, \nabla H_2\rangle \nabla H_2\} \equiv 0,$
$\|H_2\|^2 \{\|\nabla H_1\|^2 J_1 - \langle J_1, \nabla H_1\rangle \equiv 0.$

Thus either we have Case I or Case III or

$$\|H_1\|^2 J_1 - \langle J_1, H_1\rangle H_1 \equiv 0,$$

$$\|H_2\|^2 J_2 - \langle J_2, H_2\rangle H_2 \equiv 0,$$

$$\|\nabla H_1\|^2 J_1 - \langle J_1, \nabla H_1\rangle \nabla H_1 \equiv 0,$$

$$\|\nabla H_2\|^2 J_2 - \langle J_2, \nabla H_2\rangle \nabla H_2 \equiv 0.$$

That is, both M_1 and M_2 are of type AW(2) and AW(3). But we already are assuming that $\delta_1 \equiv 0$ and $\delta_2 \equiv 0$, so they also have P2-PNS and

$$\|\nabla H_1\|^2 H_1 \equiv \langle H_1, \nabla H_1\rangle \nabla H_1.$$

Substituting for $\|\nabla H_1\|^2 H_1$ this gives

$$\|\nabla H_1\|^2 \{2\langle H_1, \nabla H_1\rangle J_1 - \langle J_1, H_1\rangle \nabla H_1 - \langle J_1, \nabla H_1\rangle H_1\}$$

$$= 2\langle H_1, \nabla H_1\rangle\{\|\nabla H_1\|^2 \{\|\nabla H_1\|^2 J_1 - \langle J_1, \nabla H_1\rangle \nabla H_1\} \equiv 0$$

with a similar result for H_2, ∇H_2, J_2. Thus the conditions (h) together with $\delta_1 \equiv 0$, $\delta_2 \equiv 0$ in this case imply the condition (g).

Thus, noting that submanifolds with P2-PNS are automatically of type AW(2), we see that in this case M_1 and M_2 are both type AW(3) submanifolds with P2-PNS and this is sufficient to ensure that $M_1 \times M_2$ has P3-PNS.

Since we have now exhausted all possibilities this completes the proof of the theorem.

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