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SPHERICAL IMAGES, AND HIGHER CURVATURES

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SUMMARY

We generalize the concepts Spherical Tangent Image, Spherical principal Normal Image, Spherical Normal Image, and Spherical Indicatrix. Further, we give different expressions of higher curvatures of a curve, and present some general results about them.

ÖZET

Bu makalede Küresel Teğet Görüntüsü, Küresel Asli Normal görüntüsü, Küresel Normal Görüntü ve Küresel Gösterge kavramlarını genelleştirdik. Bundan başka bir eğrinin yüksek mertebeden eğriliklerini değişik biçimlerde ifade ettik ve onlarla ilgili olarak bazı genel sonuçlar sunduk.

O. INTRODUCTION

The spherical Images may be found in many Differential Geometry Text Books [Altın, (1979), (1986), (1987)]. The key manuscripts about the higher curvatures of a curve are given in 1966 by GLUCK. I think, the results, which we found, help us for understanding Differential Geometry with many-sided.

1. PRELIMINARIES

DEFFINITION 1.1: Let (ψ, U) be a coordinate neighborhood for the submanifold M. This means that the mapping $\psi: U \to M$ is a diffeomorhism. Then

$$\psi_\star : T_{E^r}(u) \to T_M(\psi(u))$$

is a linear transformation which corresponds to the Jacobian matrix ψ. We denote the adjoint of $\psi_\star$ by $\psi^\star$ whic is a transformation.

DEFINITION 1.2: $\psi^\star : T^\star_M(\psi(u)) \to T^\star_{E^r}(u)$

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where $T^{\star}_{M}(\psi(u))$ and $T^{\star}_{E^r}(u)$ are the dual spaces of $T_M(\psi(u))$ and $T_{E^r}(u)$, respectively. The vector spaces $T^{\star}_{M}(\psi(u))$ and $T^{\star}_{E^r}(u)$ are the cotangent spaces at the corresponding points.

DEFINITION 1.3: Let U be an Euclidean neighborhood A 1-form ω is a mapping

$$\omega : U \to \cup T^{\star}_{U}(x),\ x \in U$$

where the union is taken over all $x \in U$, such that $p \circ \omega : U \to U$ is the identity mapping

$$P : \cup T^{\star}_{U}(x) \to U,\ \ t_x \in T^{\star}_{U}(x),\ \ P(t_x) = x$$

DEFINITION 1.4: A vector field is a function

$$X : U \to \cup T_U(m)$$

Such that

$$P o X : U \to U$$

is the identity mapping, and

$$P : \cup T_U(x) \to U.\ \ P(t_x) = x,\ t_x \in T_U(x).$$

DEFINITION 1.5: Let $(x_1, ..., x_n)$ be a Euclidean coorinate system in E^n, then $[\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}]$ is a basis of the vector space x_p of all the parallel vector fields on E^n, and $[dx_1, ..., dx_n]$ is the dual basis of dual space Ω of χ_p. Let M be a r-dimensional submanifold of E^n with local coordinates $(u_1, ..., u_r)$ given by

$$x_i = x_i(u_1, ..., u_r),\quad 1 \leqslant i \leqslant n$$

Then a 1-form on M has the analytic expression

$$\omega = \sum_{i,j=1}^{n,r} \frac{\partial x_i}{\partial u_j} du_j \bigcirc \frac{\partial}{\partial x_i}$$

where $\bigcirc$ denotes the tensor product. Hence, $\omega \in T^{\star}_{U}(u) \bigcirc T_M(m)$

2. THE MAIN RESULTS

DEFINITION 2.1: If the initial points of all the unit tangent vectors X_1 of a curve α in E^n are shifted to the origin, their new end points trace out a curve $\overline{\alpha}$ (s) on the unit sphere, where $X_1 = \alpha_{\star}(\frac{\partial}{\partial s})$, and $\frac{dX_1}{ds} = dX_1(\frac{\partial}{\partial s}) \neq 0$. This new curve is called the spherical indicatrix (or spherical tangent image) of the curve α.

PROPOSITION 2.2: Let α be a curve in E^n, Let $\overline{\alpha}$ (s) be the spherical indicatrix (or spherical tangent image) of α, Let $\overline{s}$ be arc length on the spherical indicatritx, then

$$t_{12} = \frac{d\overline{s}}{ds}$$

Where s denotes arc-length on α, and t_{12} denotes the first curvature of the curve α.

PROOF: Assume that α is a regular arbitrary speedcurve in E^n, then we know that

$$\frac{ds}{dt} = \| \dot{\alpha} \|, \quad \frac{ds}{dt} = \vartheta$$

where $\dot{\alpha}$ denotes the derivative of α with respect to t. In the same way, we may write

$$\frac{d\bar{s}}{dt} = \| \dot{X}_1 \|$$

where $\dot{X}_1$ denotes the derivative of X_1 with respect to t. Now let's write the $\frac{d\bar{s}}{dt}$ detaily,

$$\frac{d\bar{s}}{dt} = \| \dot{X}_1 \|, \quad \frac{d\bar{s}}{dt} = \| \frac{dX_1}{ds} \cdot \frac{ds}{dt} \| .$$

From the equations $\frac{dX_1}{ds} = t_{12} X_2$, and $\frac{ds}{dt} = \| \dot{\alpha} \|$ we have

$$\frac{ds}{ds} = \frac{\frac{ds}{dt}}{\frac{ds}{dt}}$$

$$\frac{ds}{ds} = \frac{\| X_1 \|}{\| \alpha \|}$$

$$\frac{ds}{ds} = \frac{\| \frac{dX_1}{ds} \cdot \frac{ds}{dt} \|}{\| \alpha \|}$$

$$\frac{ds}{ds} = \frac{\| t_{12} X_2 \, \vartheta \|}{\vartheta}$$

$$\frac{ds}{ds} = | t_{12} |$$

where X_2 is the second vector of the Frenet Frame along α. Hence, we have proved our proposition.

DEFINITION 2.3: If the initial points of all the unit vectors X_i of a curve α in E^n are shifted to the origin, their new end points trace out a curve $\overset{*}{\alpha}(s)$ on the unit sphere, where X_i denotes the i^{th} orthonormal vector of the Frenet Frame along α. This new curve is called the i^{th} spherical indicatrix of the curve α.

PROPOSITION 2.4: Let α be a curve in E^n, Let $\overset{\star}{\alpha}(s)$ be the i^{th} spherical indicatrix of α, and Let $\overset{\star}{s}$ be arc length on the i^{th} spherical indicatrix, then

$$\frac{d\overset{\star}{s}}{ds} \leqslant \| t_{i-1} \| + \| t_i \|, \quad 2 \leqslant i \leqslant n-1$$

or

$$\frac{d\overset{\star}{s}}{ds} \leqslant \| t_{(i-1)i} \| + \| t_{i(i+1)} \|, \quad 2 \leqslant i \leqslant n-1$$

where s denotes arc length on α, and $t_{i(i+1)}$ denotes the i^{th} curvature of the curve α.

PROOF: Let α be a regular arbitrary speed-curve in E^n, then we may write

$$\frac{ds}{dt} = \| \alpha \| \quad \text{or} \quad \frac{ds}{dt} = \vartheta$$

where α denotes the derivative of α with respect to t. In the same way, we have

$$\frac{d\overset{\star}{s}}{dt} = \| X_i \|$$

where X_i denotes the derivative of X_i with respect to t. We can write that

$$\frac{d\overset{\star}{s}}{dt} = \| X_i \|$$

$$\frac{d\overset{\star}{s}}{dt} = \| \frac{dX_i}{ds} \cdot \frac{ds}{dt} \| .$$

From the results $\frac{dX_i}{ds} = - t_{(i-1)i} X_{i-1} + t_{i(i+1)} X_{i+1}$, and $\frac{ds}{dt} = \| \alpha \|$

we have the following.

$$\frac{d\overset{\star}{s}}{ds} = \frac{\frac{d\overset{\star}{s}}{dt}}{\frac{ds}{dt}}$$

$$\frac{d\overset{\star}{s}}{ds} = \frac{\| X_i \|}{\| \alpha \|}$$

$$\frac{d\overset{\star}{s}}{ds} = \frac{\| \frac{dX_i}{ds} \cdot \frac{ds}{dt} \|}{\| \alpha \|}$$

$$\frac{d\overset{\star}{s}}{ds} = \frac{\| - t_{(i-1)i} X_{i-1}\vartheta + t_{i(i+1)} X_{i+1}\vartheta \|}{\vartheta}$$

$$\frac{d\overset{\star}{s}}{ds} \leqslant \| t_{(i-1)i} \| + \| t_{i(i+1)} \|$$

where X_i denotes the i^{th} vector of the Frenet Frame along α. Hence the proof is completed.

DEFINITION 2.5: If the initial points of all the unit vectors X_j of a curve α in E^n are shifted to the origin, their new end points trace out a curve $\overset{\star\star}{\alpha}(s)$ on the unit sphere, where X_j denotes the j^{th} vector of the Frenet Frame along α. This new curve is called the j^{th} spherical indicatrix of the curve α.

PROPOSITION 2.6: Let α be a curve in E^n, Let $\overset{\star}{\alpha}(s)$ and $\overset{\star\star}{\alpha}(s)$ be the i^{th} and j^{th} spherical indicatrices of α, and Let $\overset{\star}{s}$ and $\overset{\star\star}{s}$ be arc lengthes on the spherical indicatrices, then,

$$\frac{d\overset{\star}{s}}{d\overset{\star\star}{s}} = \frac{\| - t_{(i-1)i} X_{i-1} + t_{i(i+1)} X_{i+1} \|}{\| - t_{(j-1)j} X_{j-1} + t_{j(j+1)} X_{j+1} \|}$$

where $t_{i(i+1)}$ and $t_{j(j+1)}$ denote the i^{th} and j^{th} curvatures of α respectively.

PROOF: Let α be a regular arbitrary speed-curve in E^n, then we may write

$$\frac{d\overset{\star}{s}}{dt} = \| \dot{X}_i \| \quad \text{and} \quad \frac{d\overset{\star\star}{s}}{dt} = \| \dot{X}_j \|$$

where $\dot{X}_i$ and $\dot{X}_j$ denote the derivatives of X_i and X_j with respect to t respectively. Now, we know that

$$\frac{d\overset{\star}{s}}{dt} = \| \dot{X}_i \| = \| \frac{dX_i}{ds} \cdot \frac{ds}{dt} \| \quad \text{and} \quad \frac{d\overset{\star\star}{s}}{dt} = \| \dot{X}_j \| = \| \frac{dX_j}{ds} \cdot \frac{ds}{dt} \| \,..$$

From the equations $\frac{dX_i}{ds} = - t_{(i-1)i} X_{i-1} + t_{i(i+1)} X_{i+1}$, and

$\frac{dX_j}{ds} = - t_{(j-1)j} X_{j-1} + t_{j(j+1)} X_{j+1}$ we obtain that

$$\frac{d\overset{\star}{s}}{d\overset{\star\star}{s}} = \frac{\dfrac{d\overset{\star}{s}}{dt}}{\dfrac{d\overset{\star\star}{s}}{dt}}$$

$$\frac{d\overset{\star}{s}}{d\overset{\star\star}{s}} = \frac{\| -t_{(i-1)i} X_{i-1} \vartheta + t_{i(i+1)} X_{i+1} \vartheta \|}{\| -t_{(j-1)j} X_{j-1} \vartheta + t_{j(j+1)} X_{j+1} \vartheta \|}$$

$$\frac{d\overset{\star}{s}}{d\overset{\star\star}{s}} = \frac{\| -t_{(i-1)i} X_{i-1} + t_{i(i+1)} X_{i+1} \|}{\| -t_{(j-1)j} X_{j-1} + t_{j(j+1)} X_{j+1} \|}$$

This result compeletes the proof of our proposition.

DEFINITION 2.7: If the initial points of all the vectors X_n of a curve α in E^n are shifted to the origin, their new end points trace out a curve $\tilde{\tilde{\alpha}}$ (s) on the unit sphere where X_n is the n^{th} vector of the Frenet Frame along α. The new curve is called the n^{th} spherical indicatrix of the curve α.

PROPOSITION 2.8: Let α be a curve in E^n, Let $\tilde{\tilde{\alpha}}$ (s) be the n^{th} spherical indicatrix of α, and let $\tilde{\tilde{s}}$ be arclength on the n^{th} spherical indicatrix then,

$$t_{(n-1)n} = \frac{d\tilde{\tilde{s}}}{ds}$$

where s denotes arc length on α, and $t_{(n-1)n}$ denotes the $(n-1)^{th}$ curvature of the curve α.

PROOF: Let α be a regular arbitrary speed-curve in E^n, then we may write

$$\frac{ds}{dt} = \vartheta \quad \text{and} \quad \frac{d\tilde{\tilde{s}}}{dt} = \| X_n \|$$

where X_n denotes the derivative of X_n with respect to t. Again we know that

$$\frac{d\tilde{\tilde{s}}}{dt} = \left\| \frac{dX_n}{ds} \cdot \frac{ds}{dt} \right\|$$

If we think the equations $\frac{dX_n}{ds} = -t_{(n-1)n} X_{n-1}$, and $\frac{ds}{dt} = \vartheta$ we see that

$$\frac{d\tilde{\tilde{s}}}{ds} = \frac{\frac{d\tilde{\tilde{s}}}{dt}}{\frac{ds}{dt}}$$

$$\frac{d\tilde{\tilde{s}}}{ds} = \frac{\| X_n \|}{\| \alpha \|}$$

$$\frac{d\tilde{\tilde{s}}}{ds} = \frac{\left\| \frac{dX_n}{ds} \cdot \frac{ds}{dt} \right\|}{\| \alpha \|}$$

$$\frac{d\tilde{\tilde{s}}}{ds} = \frac{\| t_{(n-1)n} X_{n-1} \vartheta \|}{\vartheta}, \quad \frac{d\tilde{\tilde{s}}}{ds} = t_{(n-1)n}$$

where X_{n-1} denotes the $(n-1)^{th}$ vector of the Frenet Frame along α. This completes the proof of our proposition.

PROPOSITION 2.9: Let α be a curve in E^n, let $\tilde{\alpha}$ (s) and $\overset{\star}{\alpha}$ (s) be the first and i^{th} spherical indicatrices of α, and let $\tilde{s}$ and $\overset{\star}{s}$ be arc lengthes on the spherical indicatrices, then

$$\frac{d\tilde{s}}{d\overset{\star}{s}} = \frac{\| t_{12} X_2 \|}{\| -t_{(i-1)i} X_{i-1} + t_{i(i+1)} X_{i+1} \|}$$

where t_{12} and $t_{i(i+1)}$ denote the first and i^{th} curvatures of α respectively.

PROOF: Let α be a regular arbitrary speed-curve in E^n, then we have

$$\frac{d\tilde{s}}{dt} = \| X_1 \| \quad \text{and} \quad \frac{d\overset{\star}{s}}{dt} = \| X_i \|$$

where X_1 and X_i denote the derivatives of X_1 and X_i with respect to t respectively.

Again we can write,

$$\frac{d\tilde{s}}{dt} = \| \frac{dX_1}{ds} . \frac{ds}{dt} \| \quad \text{and} \quad \frac{d\overset{\star}{s}}{dt} = \| \frac{dX_i}{ds} . \frac{ds}{dt} \| .$$

From the equations $\frac{dX_1}{ds} = t_{12} X_2$, and

$$\frac{dX_i}{ds} = - t_{(i-1)i} X_{i-1} + t_{i(i+1)} X_{i+1} \quad \text{we have}$$

$$\frac{d\tilde{s}}{d\overset{\star}{s}} = \frac{\| t_{12} X_2 \vartheta \|}{\| - t_{(i-1)i} X_{i-1} \vartheta + t_{i(i+1)} X_{i+1} \vartheta \|}$$

or

$$\frac{d\tilde{s}}{d\overset{\star}{s}} = \frac{\| t_{12} X_2 \|}{\| - t_{(i-1)i} X_{i-1} + t_{i(i+1)} X_{i+1} \|}$$

Hence, we have proved our proposition.

PROPOSITION 2.10: Let α be a curve in E^n, Let $\tilde{\alpha}$ (s) and $\tilde{\tilde{\alpha}}$ (s) be the first and n^{th} spherical indicatrices of α, and let $\tilde{s}$ and $\tilde{\tilde{s}}$ be arc lengthes on the spherical indicatrices, then

$$\frac{d\tilde{s}}{d\tilde{\tilde{s}}} = \left| \frac{t_{12}}{t_{(n-1)n}} \right|$$

where t_{12} and $t_{(n-1)n}$ denote the first and n^{th} higher curvatures of α respectively.

PROOF: Let α be a regular arbitrary speed-curve in E^n, then we obtain

$$\frac{d\tilde{s}}{dt} = \| X_1 \| \quad \text{and} \quad \frac{d\tilde{\tilde{s}}}{dt} = \| X_n \|$$

where X_1 and X_n denote the derivatives of X_1 and X_n with respect to t respectively. On the other hand we know that

$$\frac{d\tilde{s}}{dt} = \| \frac{dX_1}{ds} \cdot \frac{ds}{dt} \| \quad \text{and} \quad \frac{d\tilde{\tilde{s}}}{dt} = \| \frac{dX_n}{ds} \cdot \frac{ds}{dt} \|$$

If we think the equations $\frac{dX_1}{ds} = t_{12} X_2$, and $\frac{dX_n}{ds} = - t_{(n-1)n} X_{n-1}$ we see that

$$\frac{d\tilde{s}}{d\tilde{\tilde{s}}} = \frac{\| t_{12} X_2 \|}{\| - t_{(n-1)n} X_{n-1} \|} \quad \text{or} \quad \frac{d\tilde{s}}{d\tilde{\tilde{s}}} = \left| \frac{t_{12}}{t_{(n-1)n}} \right|$$

This result completes the proof of our proposition.

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