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THE VECTORS WHICH FORM CONSTANT ANGLES WITH THE FRENET VECTORS IN E^n

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SUMMARY

We consider about the vectors which form constant angles with the Frenet Vectors in E^n. It is the purpose of this paper to calculate these vectors.

ÖZET

E^n de Frene Vektörleri ile Sabit Açılar Yapan Vektörler

E^n de Frenet vektörleri ile sabit açılar yapan vektörler düşündük. Bu makalenin amacı bu vektörleri bulmaktır.

0. INTRODUCTION

We solved the problem in E^4. The solution and some terminology about them may be found in[1]. The concept of higher curvatures of curves in any dimensional Euclidean Space was given in[2], and[3]. In[4] it may be found rich preliminaries which we need our studies.

1. PRELIMINARIES

PROPOSITION 1.1. If the principal normals of a curve form a constant angle with the direction of a vector e, then

$$\left[\frac{\kappa^2+\tau^2}{\kappa\left(\frac{\tau}{\kappa}\right)'}\right]+\tau=0$$

conversly, if this relation is fulfilled, then the principal normals of the curve form a constant angle with the direction of some vector.

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We can express this vector by

$$e = \frac{\tau}{\kappa^2}\,\frac{\kappa^2+\tau^2}{(\frac{\tau}{\kappa})'}\,X_1 + X_2 + \frac{1}{\kappa}\,\frac{\kappa^2+\tau^2}{(\frac{\tau}{\kappa})'}\,X_3$$

This constant vector e forms with the vector X_2 a constant angle whose cosine equals $\frac{1}{|e|}$ = constant.

2. THE MAIN RESULTS

PROPOSITION 2.1. If the first principal vectors of a curve form a constant angle with the direction of a vector γ, then

$$\left[\frac{1}{t_{(n-1)n}}\left(t_{(n-2)(n-1)}\alpha_{n-2} + \alpha'_{n-1}\right)\right]' + \frac{t_{(n-1)n}}{t_{(n-2)(n-1)}}\left(t_{(n-3)(n-2)}\alpha_{n-3} + \alpha'_{n-2}\right) = 0$$

Conversly, if this relation is fulfilled, then the first principal vectors of the curve form a constant angle with the direction of some vector. Further, we may write this vector and the angle as the following.

$$\gamma = X_1 + \frac{t_{12}}{t_{23}}X_3 + \frac{1}{t_{34}}\left(\frac{t_{12}}{t_{23}}\right)'X_4 + \frac{1}{t_{45}}\left[\frac{t_{12}\,t_{34}}{t_{23}} + \left\{\frac{1}{t_{34}}\left(\frac{t_{12}}{t_{23}}\right)'\right\}'\right]X_5$$

$$+ \ldots\ldots + \frac{1}{t_{(n-2)(n-1)}}\left(t_{(n-3)(n-2)}\alpha_{n-3} + \alpha'_{n-2}\right)X_{n-1} + \frac{1}{t_{(n-1)n}}.$$

$$\left(t_{(n-2)(n-1)}\alpha_{n-2} + \alpha'_{n-1}\right)X_n.$$

$$\mathrm{Cos}\theta = \frac{1}{|\gamma|} = \mathrm{Constant}$$

where θ is the angle between the principal vector X_1 and γ

PROOF. From

$$<\gamma, X_1> = C$$

by differentiating, we have

$$t_{12} <\gamma, X_2> = 0$$

or

$$<\gamma, X_2> = 0$$

where C is a real number.

In the same way, we obtain

$$-t_{12}<\gamma, X_1>+t_{23}<\gamma, X_3>=0$$

or

$$<\gamma, X_3>=C.\frac{t_{12}}{t_{23}}.$$

Differentiating again, we find

$$-t_{23}<\gamma, X_2>+t_{34}<\gamma, X_4>=C.(\frac{t_{12}}{t_{23}})'$$

$$<\gamma, X_4>=C.\frac{1}{t_{34}}(\frac{t_{12}}{t_{23}})'.$$

Differentiating once again, we have

$$-t_{34}<\gamma, X_3>+t_{45}<\gamma, X_5>=C.\left\{\frac{1}{t_{34}}(\frac{t_{12}}{t_{23}})'\right\}'$$

or

$$<\gamma, X_5>=\frac{C}{t_{45}}\left[\frac{t_{12}\ t_{34}}{t_{23}}+\left\{\frac{1}{t_{34}}(\frac{t_{12}}{t_{23}})'\right\}'\right]$$

....

From the above, we can write

$$<\gamma, X_{n-1}>=\frac{C}{t_{(n-2)\ (n-1)}}-(t_{(n-3)\ (n-2)}\alpha_{n-3}+\alpha'_{n-2})$$

$$<\gamma, X_n>=\frac{C}{t_{(n-1)n}}(t_{(n-2)\ (n-1)}\alpha_{n-2}+\alpha'_{n-1})$$

Finally, differentiating, we obtain

$$\left[\frac{1}{t_{(n-1)n}}(t_{(n-2)\ (n-1)}\alpha_{n-2}+\alpha'_{n-1})\right]'+$$

$$\frac{t_{(n-1)n}}{t_{(n-2)\ (n-1)}}(t_{(n-3)\ (n-2)}\alpha_{n-3}+\alpha'_{n-2})=0$$

Conversly, if the relation is held, then this vector is constant. The constant vector γ forms with the principal vector X_1 an angle whose cosine equals $\frac{1}{|\gamma|}=$ constant. Without loss generality, we assumed that $C=1$. Hence, the proposition is proved.

PROPOSITION 2.2. If the n^{th} Frenet Vectors of a curve in E^n form a constant angle with the direction of a vector ξ, then

$$\left[\frac{1}{t_{12}}(t_{23}\alpha_3+\alpha'_2)\right]'-\frac{t_{12}}{t_{23}}(t_{34}\alpha_4+\alpha'_3)=0.$$

Conversly, if this relation is satisfied, then the n^{th} Frenet Vectors of the curve form a constant angle with the direction of some vector. We may express this vector and the angle as the following

$$\xi = \frac{1}{t_{12}}(t_{23}\,\alpha_3 + \alpha'_2)\,X_1 + \frac{1}{t_{23}}(t_{34}\,\alpha_4 + \alpha'_3)\,X_2 + \dots\dots\dots$$

$$+ \frac{1}{t_{(n-4)(n-3)}}\left\{\left[\frac{t_{(n-3)(n-2)}\,t_{(n-1)n}}{t_{(n-2)(n-1)}} + \left[\frac{1}{t_{(n-3)(n-2)}}\left(\frac{t_{(n-1)n}}{t_{(n-2)(n-1)}}\right)'\right]'\right]'\right\}X_{n-4} -$$

$$\frac{1}{t_{(n-3)(n-2)}}\left(\frac{t_{(n-1)n}}{t_{(n-2)(n-1)}}\right)' X_{n-3} + \frac{t_{(n-1)n}}{t_{(n-2)(n-1)}} X_{n-2} + X_n \,.$$

$$\cos\varphi = \frac{1}{|\,\xi\,|} = \text{Constant}$$

where φ is the angle between X_n and ξ.

PROOF. Let

$$<\xi, X_n> = C$$

where C is a real number. Then, we hand

$$-t_{(n-1)n} <\xi, X_{n-1}> = 0$$

or

$$<\xi, X_{n-1}> = 0$$

By differentiating, we obtain

$$-t_{(n-2)(n-1)} <\xi, X_{n-2}> + t_{(n-1)n} <\xi, X_n> = 0$$

or

$$<\xi, X_{n-2}> = C \,.\, \frac{t_{(n-1)n}}{t_{(n-2)(n-1)}}$$

Differentiating again, we have

$$-t_{(n-3)(n-2)} <\xi, X_{n-3}> + t_{(n-2)(n-1)} <\xi, X_{n-1}> = C\,.\left(\frac{t_{(n-1)n}}{t_{(n-2)(n-1)}}\right)'$$

$$<\xi, X_{n-3}> = -\frac{C}{t_{(n-3)(n-2)}}\left(\frac{t_{(n-1)n}}{t_{(n-2)(n-1)}}\right)' \,.$$

Differentiating once again, we hand

$$-t_{(n-4)(n-3)} < \xi, X_{n-4} > + t_{(n-3)(n-2)} < \xi, X_{n-2} > =$$

$$-C \cdot \left[\frac{1}{t_{(n-3)(n-2)}} \left(\frac{t_{(n-1)n}}{t_{(n-2)(n-1)}} \right)' \right]'$$

or

$$< \xi, X_{n-4} > = \frac{C}{t_{(n-4)(n-3)}} \left[\frac{t_{(n-3)(n-2)} t_{(n-1)n}}{t_{(n-2)(n-1)}} + \right.$$

$$\left. \left[\frac{1}{t_{(n-3)(n-2)}} \left(\frac{t_{(n-1)n}}{t_{(n-2)(n-1)}} \right)' \right]' \right]$$

From the above results, we may write

$$< \xi, X_2 > = \frac{C}{t_{23}} (t_{34} \alpha_4 + \alpha'_3)$$

$$< \xi, X_1 > = \frac{C}{t_{12}} (t_{23} \alpha_3 + \alpha'_2).$$

Differentating the last equation, we find

$$t_{12} < \xi, X_2 > = C \cdot \left[\frac{1}{t_{12}} (t_{23} \alpha_3 + \alpha'_2) \right]'$$

or

$$\left[\frac{1}{t_{12}} (t_{23} \alpha_3 + \alpha'_2) \right]' - \frac{t_{12}}{t_{23}} (t_{34} \alpha_4 + \alpha'_3) = 0.$$

Conversly, if the relation is fulfilled, then this vector is constant. This constant vector ξ forms with the vector X_n an angle whose cosine equals $\frac{1}{|\xi|}$ = constant. Without loss generality, again we assumed that C = 1. Hence, we have completed the proof of our proposition.

PROPOSITION 2.3. If the second Frenet vectors of a curve in E^4 form a constant angle with the direction of a vector §, then

$$\left[\frac{\left(\frac{t_{23}}{t_{12}}\right)' \left[\left(t_{12} + \frac{t_{23}^2}{t_{12}}\right)' + t_{23} \left(\frac{t_{23}}{t_{12}}\right)' \right] + \left(t_{12} + \frac{t_{23}^2}{t_{12}}\right) \left[\frac{t_{23} t_{34}^2}{t_{12}} - \left(\frac{t_{23}}{t_{12}}\right)'' \right]}{\left(\frac{t_{23}}{t_{12}}\right)' \left[t_{34} \left(\frac{t_{23}}{t_{12}}\right)' + \left(\frac{t_{23} t_{34}}{t_{12}}\right)' \right] + \frac{t_{23} t_{34}}{t_{12}} \left[\frac{t_{23} t_{34}^2}{t_{12}} - \left(\frac{t_{23}}{t_{12}}\right)'' \right]} \right]'$$

$$+t_{34}\frac{(t_{12}+\frac{t_{23}^2}{t_{12}})\left\{t_{34}(\frac{t_{23}}{t_{12}})'+(\frac{t_{23}t_{34}}{t_{12}})'\right]-\frac{t_{23}t_{34}}{t_{12}}\left[(t_{12}+\frac{t_{23}^2}{t_{12}})'+t_{23}(\frac{t_{23}}{t_{12}})'\right]}{(\frac{t_{23}}{t_{12}})'\left\{t_{34}(\frac{t_{23}}{t_{12}})'(\frac{t_{23}t_{34}}{t_{12}})'\right]+\frac{t_{23}t_{34}}{t_{12}}\left\{\frac{t_{23}t_{34}^2}{t_{12}}-(\frac{t_{23}}{t_{12}})''\right]}=0$$

Conversly, if this relation is fulfilled, then the second Frenet Vectors of the curve form a constant angle with the direction of some vector. We may write this vector and the angle as the following

$$\S=\frac{\frac{t_{23}}{t_{12}}(t_{12}+\frac{t_{23}^2}{t_{12}})\left\{t_{34}(\frac{t_{23}}{t_{12}})'+(\frac{t_{23}t_{34}}{t_{12}})'\right]-\frac{t_{23}t_{34}}{t_{12}}\left\{(t_{12}+\frac{t_{23}^2}{t_{12}})'+t_{23}(\frac{t_{23}}{t_{12}})'\right]}{(\frac{t_{23}}{t_{12}})'\left[t_{34}(\frac{t_{23}}{t_{12}})'+(\frac{t_{23}t_{34}}{t_{12}})'\right]+\frac{t_{23}t_{34}}{t_{12}}\left\{\frac{t_{23}t_{34}^2}{t_{12}}-(\frac{t_{23}}{t_{12}})''\right]}X_1+$$

$$X_2+\frac{(t_{12}+\frac{t_{23}^2}{t_{12}})\left[t_{34}(\frac{t_{23}}{t_{12}})'+(\frac{t_{23}t_{34}}{t_{12}})'\right]-\frac{t_{23}t_{34}}{t_{12}}\left[(t_{12}+\frac{t_{23}^2}{t_{12}})'+t_{23}(\frac{t_{23}}{t_{12}})'\right]}{(\frac{t_{23}}{t_{12}})'\left\{t_{34}(\frac{t_{23}}{t_{12}})'+(\frac{t_{23}t_{34}}{t_{12}})'\right]+\frac{t_{23}t_{34}}{t_{12}}\left\{\frac{t_{23}t_{34}^2}{t_{12}}-(\frac{t_{23}}{t_{12}})''\right]}X_3$$

$$+\frac{(\frac{t_{23}}{t_{12}})'\left[t_{12}+\frac{t_{23}^2}{t_{12}})'+t_{23}(\frac{t_{23}}{t_{12}})'\right]+(t_{12}+\frac{t_{23}^2}{t_{12}})\left\{\frac{t_{23}t_{34}^2}{t_{12}}-(\frac{t_{23}}{t_{12}})''\right]}{(\frac{t_{23}}{t_{12}})'\left[t_{34}(\frac{t_{23}}{t_{12}})'+(\frac{t_{23}t_{34}}{t_{12}})'\right]+\frac{t_{23}t_{34}}{t_{12}}\left[\frac{t_{23}t_{34}^2}{t_{12}}-(\frac{t_{23}}{t_{12}})''\right]}X_4$$

$$\text{Cos}\,\upsilon=\frac{1}{|\S|}=\text{Constant}$$

where υ is the angle between X_2 and $\S$.

PROOF. Take

$$<\S,X_2>=C.$$

Differentiating, we hand

$$-t_{12}<\S,X_1>+t_{23}<\S,X_3>=0$$

$$<\S,X_1>=\frac{t_{23}}{t_{12}}<\S,X_3>.$$

Differentiating again, we find

$$Ct_{12}=(\frac{t_{23}}{t_{12}})'<\S,X_3>-C\frac{t_{23}^2}{t_{12}}+\frac{t_{23}t_{34}}{t_{12}}<\S,X_4>$$

or

$$(\frac{t_{23}}{t_{12}})' < §, X_3 > + \frac{t_{23}t_{34}}{t_{12}} < §, X_4 > = C\,(t_{12} + \frac{t_{23}^2}{t_{12}})$$

Differentiating once again, we have

$$(\frac{t_{23}}{t_{12}})'' < §, X_3 > - C\,(\frac{t_{23}}{t_{12}})'\, t_{23} + t_{34}\,(\frac{t_{23}}{t_{12}})' < §, X_4 > +$$

$$(\frac{t_{23}t_{34}}{t_{12}})' < §, X_4 > - \frac{t_{23}t_{34}^2}{t_{12}} < §, X_3 > = C\,(t_{12} + \frac{t_{23}^2}{t_{12}})' \quad \text{or}$$

$$(\frac{t_{23}}{t_{12}})'' - \frac{t_{23}t_{34}^2}{t_{12}} < §, X_3 > + \quad t_{34}\,(\frac{t_{23}}{t_{12}})' + (\frac{t_{23}t_{34}}{t_{12}})' < §, X_4 > =$$

$$C\left[(t_{12} + \frac{t_{23}^2}{t_{12}})' + t_{23}\,(\frac{t_{23}}{t_{12}})'\right].$$

Now we can write the following equations

$$D = \begin{bmatrix} 1 & -\frac{t_{23}}{t_{12}} & 0 \\ 0 & (\frac{t_{23}}{t_{12}})' & \frac{t_{23}t_{34}}{t_{12}} \\ 0 & (\frac{t_{23}}{t_{12}})'' - \frac{t_{23}t_{34}^2}{t_{12}} & t_{34}\,(\frac{t_{23}}{t_{12}})' + (\frac{t_{23}t_{34}}{t_{12}})' \end{bmatrix}$$

$$D < §, X_1 > = \begin{bmatrix} 0 & -\frac{t_{23}}{t_{12}} & 0 \\ C\,(t_{12} + \frac{t_{23}^2}{t_{12}}) & (\frac{t_{23}}{t_{12}})' & \frac{t_{23}t_{34}}{t_{12}} \\ C\left[(t_{12} + \frac{t_{23}^2}{t_{12}})' + t_{23}\,(\frac{t_{23}}{t_{12}})'\right] & (\frac{t_{23}}{t_{12}})'' - \frac{t_{23}t_{34}^2}{t_{12}} & t_{34}\,(\frac{t_{23}}{t_{12}})' + (\frac{t_{23}t_{34}}{t_{12}})' \end{bmatrix}$$

$$D < §, X_3 > = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\,(t_{12} + \frac{t_{23}^2}{t_{12}}) & \frac{t_{23}t_{34}}{t_{12}} \\ 0 & C\left[(t_{12} + \frac{t_{23}^2}{t_{12}})' + t_{23}\,(\frac{t_{23}}{t_{12}})'\right] & t_{34}\,(\frac{t_{23}}{t_{12}})' + (\frac{t_{23}t_{34}}{t_{12}})' \end{bmatrix}$$

$$D<\S, X_4> = \begin{bmatrix} 1 & -\frac{t_{23}}{t_{12}} & 0 \\ 0 & (\frac{t_{23}}{t_{12}})' & C\,(t_{12} + \frac{t_{23}^2}{t_{12}}) \\ 0 & (\frac{t_{23}}{t_{12}})'' - \frac{t_{23}\, t_{34}^2}{t_{12}} & C\left[(t_{12} + \frac{t_{23}^2}{t_{12}})' + t_{23}\,(\frac{t_{23}}{t_{12}})'\right] \end{bmatrix}$$

Conversly, if the relation is fulfilled, then this vector is constant. This constant vector § forms with the X_2 an angle whose cosine equals $\frac{1}{|\S|}$ = constant. Without loss generality, we may assume that $C = 1$. Hence we have proved the proposition.

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