Uludağ Üniversitesi
Eğitim Fakülteleri Dergisi
Cilt: V, Sayı: 2, 1990

A General Relation Between Numbers of the Spanning Trees of Graphs B_n and $L(B_n)$

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ABSTRACT

The definitions of the adjacency matrices A, A_L and the incidence matrices D, D_L of the graphs B_n, $L(B_n)$ and the relation between the characteristic polynomials of these graphs are presented in [1]. In this study, two theorems which shows the relationship between numbers of the spanning trees of the graphs B_n and $L(B_n)$ are proved. The results are discussed by applying these theorems to the graphs B_1, B_2, B_3 and its line graphs $L(B_1)$, $L(B_2)$, $L(B_3)$.

ÖZET

B_n ve $L(B_n)$ Graflarının Kapsar Ağaçlarının Sayıları Arasında Genel Bir İlişki

B_n, $L(B_n)$ graflarının A, A_L bağlantı matrislerinin ve D, D_L değme matrislerinin tanımları ve bu grafların karakteristik polinomları arasındaki ilişki [1] de ortaya konulmuştur. Bu çalışmada, B_n ve $L(B_n)$ graflarının kapsar ağaçlarının sayıları arasında ilişkiyi sergileyen iki teorem ispatlanmıştır. Bu teoremler B_1, B_2, B_3 grafları ve bunların $L(B_1)$, $L(B_2)$, $L(B_3)$ ayrıt graflarına uygulanarak sonuçlar tartışılmıştır.

1. INTRODUCTION

The problem of finding bases for the circuit - and cutset - subspaces is of great practical and theoretical importance in electrical network analysis. This problem was originally solved by Kirchhoff [5]. Spanning trees of the graph

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which was corresponded to an electrical network played an importance role in this solution.

DEFINITION 1.1: Let G be a connected graph with v vertices and e edges and $G_a$be a connected subgraph of G. If G_a contains all the vertices of G but does not contain any circuits of G, G_a is called a spanning tree of graph G.

The fundamental properties of the spannin tree which can be got directly from definition 1.1 are given below.

PROPERTY 1.1: There is one and only one path between any two vertices of a spanning tree.

PROPERTY 1.2: A spanning tree of a connected graph with v vertices has v-1 branches [3, p. 114].

PROPERTY 1.3: Let G be a connected graph v vertices and e edges and let G_a be a spanning tree of G and let $\overline{G}_a$be a subgraph which is also complement of Ga. Them the number of chords in G_a is $e-v+1$ [3, p. 114].

Generally, a lot of different spanning trees can be selected from a connected graph G with v vertices. The circuit and cutset subspaces of the graph G can be formed according to the selected spanning tree.

DEFINITION 1.2: The number of the spanning trees of the connected and linear graph G with v vertices is called complexity of this graph and this number is denoted by k(G). If the graph G is disconnected, then $k(G) = 0$.

The four propositions which are well-known about the number k(G) are explained below.

PROPOSITION 1.1: A matrix called M is obtained by putting n instead of a_{ii} which are entries on the main diagonal of the adjacency matrix A of B_n and by putting $-a_{ij}$ instead of entries a_{ij} of A for $i \neq j$. If the degree matrix of B_n is denoted by Δ, then $M = \Delta - A$. The cofactors of M is equal to same number and this number is the number of the spanning trees of B_n [4, p. 153-155].

As the number of spanning trees of B_n can be found like in Proposition 1.1 by using the adjacency matrix A, it can also be found like in Proposition 1.2 by using incidence matrix $D = [d_{ij}]_{vxe}$.

PROPOSITION 1.2: Let D be the incidence matrix of a connected and linear graph G with v vertices and e edges. Then the cofactors of DD^t is equal to a same number and this number is the number of the spanning trees of B_n [5].

PROPOSITION 1.3: The number of spanning trees in the complete graph with n vertices is n^{n-2} [2].

PROPOSITION 1.4: The number of spanning trees in the connected and simple graph G(v, e) is $k(G) = v^{-2} \det (J+M)$ [6].

The matrix M in this proposition is explained in proposition 1.1 and the matrix J is a matrix which is in the same dimension with M and all of whose entries are 1.

2. MATERIAL AND METHOD

THEOREM 2.1: It was pointed out in [1, p. 8] that the spectrum of graph B_n which is a regular graph of the n^{th} degree and which has $v = 2^n$ vertices is

$$\text{Spec } B_n = \begin{pmatrix} n & \lambda_1 & \lambda_2 \dots \lambda_s \\ 1 & \varsigma_1 & \varsigma_2 \dots \varsigma_s \end{pmatrix}$$

Under this given hypothesis, the number of spanning trees of graph B_n is found by formula

$$k(B_n) = v^{-1} \prod_{r=1}^{s} (n - \lambda_r)^{\varsigma_r} = v^{-1} K'_{B_n}(n).$$

$K'_{B_n}(n)$ in this formula is obtained by putting n instead of λ in the derivative of the characteristic polynomial of B_n.

PROOF: The equality of

$$J + M = J + \Delta - A = J + nI - A \tag{1}$$

is obtined by putting $\Delta - A$, which is pointed out in Proposition 1.1, instead of the matrix M and by considering that the graph B_n is a regular graph of the n^{th} degree. Since the matrix J commutes with the matrix A ($AJ = JA = nJ$) and the eigenvalues of the matrix J are $v = 2^n$ (once) and 0 ($2^n - 1$ times), by considering that the hypothesis and the equality (1), it follows that the eigenvalues of matrix $J + nI - A$ are $v = 2^n$ (once) and $n - \lambda_r$ (ς_r times) for $1 \leq r \leq s$. The determinant of matrix $J+M = J+nI-A$ whose eigenvalues and their multiplicities are known is

$$\det(J + M) = v \prod_{r=1}^{s} (n - \lambda_r)^{\varsigma_r}. \tag{2}$$

By writing the value which is pointed out in Proposition 1.4 instead of the determinant on the left side of the equality (2) we have

$$v^2 k(B_n) = v \prod_{r=1}^{s} (n - \lambda_r)^{\varsigma_r}$$

or

$$k(B_n) = v^{-1} \prod_{r=1}^{s} (n - \lambda_r)^{\varsigma_r}. \tag{3}$$

Hence, we have proved first part of this theorem.

Since B_n is a regular graph of n^{th} degree and n is the largest eigenvalue of B_n so that the multiplicity of n is one [1, p. 8], $\lambda - n$ is a simple factor of characteristic polynomial $K_{B_n}(\lambda)$ of the graph B_n. The characteristic polynomial $K_{B_n}(\lambda)$ of B_n which has 2^n vertices is a polynomial of $(2^n)^{th}$ degree according to λ. Since $\lambda - n$ is a simple factor of this polynomial we can write

$$K_{B_n}(\lambda) = (\lambda - n)\, f(\lambda) \tag{4}$$

Where $f(\lambda)$ is a polynomial of $(2^{n-1})^{th}$ degree according to λ. By using the law of the derivative of a product in (4) we obtoin

$$K'_{B_n}(\lambda) = f(\lambda) + (\lambda - n)\, f'(\lambda). \tag{5}$$

By writing n istead of λ in (5) we get

$$K'_{B_n}(n) = f(n). \tag{6}$$

Since

$$\text{Spec}\, B_n = \begin{pmatrix} n & \lambda_1 & \lambda_2 \ldots\ldots & \lambda_s \\ 1 & \varsigma_1 & \varsigma_2 \ldots\ldots & \varsigma_s \end{pmatrix}$$

according to hypothesis, the characteristic polynomial of B_n is

$$K_{B_n}(\lambda) = (\lambda - n) \prod_{r=1}^{s} (\lambda - \lambda_r)^{\varsigma_r}. \tag{7}$$

where $\varsigma_1 + \varsigma_2 + \ldots + \varsigma_s = 2^{n-1}$. From the equalities (4) and (7) we obtain

$$f(\lambda) = \prod_{r=1}^{s} (\lambda - \lambda_r)^{\varsigma_r}. \tag{8}$$

By writing n instead of λ in (8) we get

$$f(n) = \prod_{r=1}^{s} (n - \lambda_r)^{\varsigma_r}. \tag{9}$$

From the equalities (6) and (9) we have

$$K'_{B_n}(n) = \prod_{r=1}^{s} (n - \lambda_r)^{\varsigma_r} \tag{10}$$

We can also find the result

$$k(B_n) = v^{-1} \prod_{r=1}^{s} (n - \lambda_r) = v^{-1}\, K'_{B_n}(n) \tag{11}$$

from (3) and (10). The result completes the proof of the Theorem 2.1.

THEOREM 2.2: If the numbers of spanning trees of the graph B_n which has $v = 2^n$ vertices and $e = n2^{n-1}$ edges and which is a regular graph of n^{th} degree and its line graph $L(B_n)$ is $k(B_n)$ and $k(L(B_n))$ respectively, then there is a relation of

$$k(L(B_n)) = 2^{e-v+1}\, n^{e-v-1}\, k(B_n)$$

between these numbers.

PROOF: There is a relation of

$$2e = nv \tag{12}$$

between the number of vertices and the number of edges of the graph B_n which has $v = 2^n$ vertices and $e = n2^{n-1}$ edges. The line graph $L(B_n)$ of the graph B_n is constructed by taking the edges of B_n as vertices of $L(B_n)$, and joining two vertices in $L(B_n)$ whenever the corresponding edges in B_n have a common vertex. According to this construction, the line graph $L(B_n)$ is a regular graph of $(2n-2)^{th}$ degree and it has $e= n2^{n-1}$ vertices and $u= n(n-1)2^{n-1}$ edges. It was pointed out in [1, p. 8] that the spectrum of the graph B_n was

$$\operatorname{Spec} L(B_n) = \begin{pmatrix} 2n-2 & n-2+\lambda_1 & n-2+\lambda_2 & \dots & -2 \\ 1 & \varsigma_1 & \varsigma_2 & \dots & e-v \end{pmatrix}$$

From here the graph $L(B_n)$ supplies the hypothesis in Theorem 2.1. By applying Theorem 2.1 to the graph $L(B_n)$ we obtain

$$k(L(B_n)) = e^{-1} K'_{L(B_n)}(2n-2). \tag{13}$$

The relation of

$$K_{L(B_n)}(\lambda) = (\lambda+2)^{e-v} K_{B_n}(\lambda+2-n) \tag{14}$$

between the characteristic polynomials of the graphs B_n and $L(B_n)$ was proved in [1, p. 4-5). By taking the derivative of both sides of the equality (14) we obtain

$$K'_{L(B_n)}(\lambda) = (e-v)(\lambda+2)^{e-v-1} K_{B_n}(\lambda+2-n)+(\lambda+2)^{e-v} K'_{B_n}(\lambda+2-n) \tag{15}$$

By putting 2n-2 instead of λ in (15) we get

$$K'_{L(B_n)}(2n-2) = (e-v)(2n)^{e-v-1} K_{B_n}(n) + (2n)^{e-v} K'_{B_n}(n) \tag{16}$$

It is seen that

$$K_{B_n}(n) = 0 \tag{17}$$

if we write $\lambda = n$ in (4). From (17) and (16) it is found out that,

$$K'_{L(B_n)}(2n-2) = 2^{e-v} n^{e-v} K'_{B_n}(n) \tag{18}$$

From (18) and (13) we get

$$k(L(B_n)) = e^{-1} 2^{e-v} n^{e-v} K'_{B_n}(n). \tag{19}$$

By writing $v\,k(B_n)$ in (11) instead of $K'_{B_n}(n)$ in (19) and by considering the equality in (12) we have

$$k(L(B_n)) = 2^{e-v+1} n^{e-v-1} k(B_n) \tag{20}$$

as a result.

3. RESULT AND DISCUSSION

The number $k(B_n)$ of the spanning trees of the graph B_n is found by the help of Theorem 2.1 or Proposition 1.1. The number $k(L(B_n)$ of the spanning trees of the line graph $L(B_n)$ of B_n is found by the help of Theorem 2.2. The formula (20) which was proved in Theorem 2.2 is the formula that helps to find $k(L(B_n))$ while $k(B_n)$ is known.

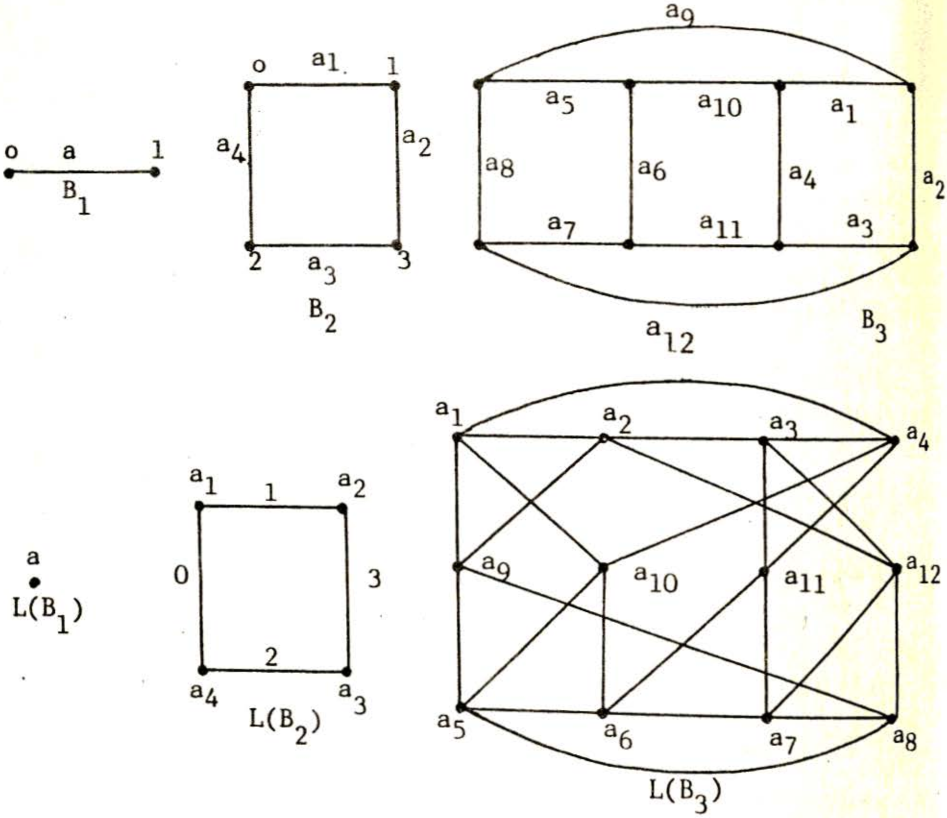

Figure: 1
Graphs B_1, B_2, B_3 and its line graphs $L(B_1)$, $L(B_2)$, $L(B_3)$

For example, the spectrums of graphs B_1, B_2, B_3 are in order

$$\text{Spec } B_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \text{Spec } B_2 = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 2 & 1 \end{pmatrix}, \text{Spec } B_3 = \begin{pmatrix} 3 & 1 & -1 & -3 \\ 1 & 3 & 3 & 1 \end{pmatrix}. \quad (21)$$

From (21) and (11) it is found that

$$k(B_1) = 1, \quad k(B_2) = 4, \quad k(B_3) = 384 \qquad (22)$$

The result

$$k(L(B_1)) = 1, \quad k(L(B_2)) = 4, \quad k(L(B_3)) = 331776$$

is found if we write the values found in their place in formula (20).

By applying Theorem 2.1 and Theorem 2.2 to the graphs B_4, B_5, ..., B_n ($n \geq 4$) and their line graphs $L(B_4)$, $L(B_5)$, ..., $L(B_n)$ like in the example above, we come to a result that the numbers $k(L(B_4))$, $k(L(B_5))$, ..., $k(L(B_n))$ can be found while $k(B_4)$, $k(B_5)$, ..., $k(B_n)$ is known.

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